

ASYMPTOTIC REGULARITY AND FIXED POINT THEOREMS ON A 2-BANACH SPACE

Mantu Saha, Debashis Dey, Anamika Ganguly and Lokenath Debnath

Abstract. The present paper deals with some fixed point theorems for a class of mixed type of contraction maps possessing the asymptotically regular property in a 2-Banach space.

1 Introduction

Considerable attention has been given to fixed points and fixed point theorems in metric and Banach spaces due to their tremendous applications to mathematics. Motivated by this work, several authors introduced similar concepts and proved analogous fixed point theorems in 2-metric and 2-Banach spaces as cited in the papers of the following authors. Gähler ([3], [4]) investigated the idea of 2-metric and 2-Banach spaces and proved some results. Subsequently, several authors including Iseki [5], Rhoades [7], and White [8] studied various aspects of the fixed point theory and proved fixed point theorems in 2-metric spaces and 2-Banach spaces. On the other hand, Cho et al. [2] investigated common fixed points of weakly commuting mappings and examined the asymptotic regular property in 2-metric space. Panja and Baisnab [6] studied asymptotic regularity and common fixed point theorems. In spite of the above work, the asymptotic regularity and fixed point theorems on a 2-Banach space need more investigation. So, the major objective of this paper is to study some fixed point theorems for a class of mixed type of contraction mappings possessing the asymptotically regular property in a 2-Banach space.

2 Preliminaries

Definition 1. Let X be a real linear space and $\|.,.\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions :

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent in X ,

2010 Mathematics Subject Classification: 47H10; 54H25.

Keywords: Asymptotically regular; 2-normed space 2-Banach space; fixed points and fixed point theorems.

<http://www.utgjiu.ro/math/sma>

- (ii) $\|x, y\| = \|y, x\|$, for all $x, y \in X$,
- (iii) $\|x, ay\| = |a| \|x, y\|$, a being real, $x, y \in X$
- (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all $x, y, z \in X$

Then $\|\cdot, \cdot\|$ is called a 2 - norm and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and all real numbers a .

Definition 2. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_m - x_n, y\| = 0$ for all y in X .

Definition 3. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot, \cdot\|)$ is said to be convergent if there is a point x in X such that $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$ for all y in X . If $\{x_n\}$ converges to x , we write $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$.

Definition 4. A linear 2 - normed space X is said to be complete if every Cauchy sequence is convergent to an element of X . We then call X to be a 2-Banach space.

Definition 5. Let X be a 2-Banach space and T be a self-mapping of X . T is said to be continuous at x if for every sequence $\{x_n\}$ in X , $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$ implies $\{T(x_n)\} \rightarrow T(x)$ as $n \rightarrow \infty$

We cite some examples of 2-Banach spaces from the current literature (see [1], [8]).

Example 6. Let X is R^3 and consider the following 2-norm on X as

$$\|x, y\| = \left| \det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|, \text{ where } x = (x_1, x_2, x_3) \text{ and } y = (y_1, y_2, y_3). \text{ Then } (X, \|\cdot, \cdot\|) \text{ is a 2-Banach space.}$$

Example 7. Let P_n denotes the set of all real polynomials of degree $\leq n$, on the interval $[0, 1]$. By considering usual addition and scalar multiplication, P_n is a linear vector space over the reals. Let $\{x_0, x_1, \dots, x_{2n}\}$ be distinct fixed points in $[0, 1]$ and define the following 2-norm on P_n :

$$\|f, g\| = \sum_{k=0}^{2n} |f(x_k)g(x_k)|, \text{ whenever } f \text{ and } g \text{ are linearly independent}$$

and $\|f, g\| = 0$, if f, g are linearly dependent.

Then $(P_n, \|\cdot, \cdot\|)$ is a 2-Banach space.

Example 8. Let X is Q^3 , the field of rational number and consider the following 2-norm on X as:

$\|x, y\| = \left| \det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \|\cdot, \cdot\|)$ is not a 2-Banach space but a 2-normed space.

3 Main Results

First of all we give a definition of asymptotic regularity in a 2-normed linear space.

Definition 9. (Asymptotic regularity in 2-normed linear space) Let $(X, \|\cdot, \cdot\|)$ be a 2-normed linear space with 2-norm $\|\cdot, \cdot\|$. A mapping T of X into itself is said to be asymptotically regular (briefly a.r.) at some point x in X if $\lim_n \|T^n(x) - T^{n+1}(x), a\| = 0$ for all $a \in X$, where $T^n(x)$ denotes the n -th iterate of T at x .

Theorem 10. Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and T be a mapping of X into itself such that for every $x, y, a \in X$

$$\begin{aligned} \|T(x) - T(y), a\| &\leq \alpha [\|x - T(x), a\| + \|y - T(y), a\|] + \beta \|x - y, a\| \\ &\quad + \gamma \max \{ \|x - T(y), a\|, \|y - T(x), a\| \} \end{aligned} \quad (3.1)$$

where $\alpha, \beta, \gamma \geq 0$ are such that $\max \{ \alpha, \beta \} + \gamma < \frac{1}{2}$. Then T has a unique fixed point in X if T is asymptotically regular (a.r.) at some point in X .

Proof. Let T be asymptotically regular at $x_0 \in X$. Then for positive integers m, n ;

$$\begin{aligned} \|T^m(x_0) - T^n(x_0), a\| &= \|T(T^{m-1}(x_0)) - T(T^{n-1}(x_0)), a\| \\ &\leq \alpha [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^{n-1}(x_0) - T^n(x_0), a\|] \\ &\quad + \beta \|T^{m-1}(x_0) - T^{n-1}(x_0), a\| \\ &\quad + \gamma \max \{ \|T^{m-1}(x_0) - T^m(x_0), a\|, \\ &\quad \|T^{n-1}(x_0) - T^n(x_0), a\| \} \\ &\leq \alpha [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^{n-1}(x_0) - T^n(x_0), a\|] \\ &\quad + \beta \|T^{m-1}(x_0) - T^{n-1}(x_0), a\| \\ &\quad + \gamma [\|T^{m-1}(x_0) - T^m(x_0), a\| \\ &\quad + \|T^{n-1}(x_0) - T^n(x_0), a\|] \\ &= \alpha [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^{n-1}(x_0) - T^n(x_0), a\|] \\ &\quad + \beta [\|T^{m-1}(x_0) - T^m(x_0), a\| \\ &\quad + \|T^m(x_0) - T^n(x_0), a\| + \|T^n(x_0) - T^{n-1}(x_0), a\|] \\ &\quad + \gamma [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^m(x_0) - T^n(x_0), a\| \\ &\quad + \|T^{n-1}(x_0) - T^n(x_0), a\| + \|T^n(x_0) - T^{n-1}(x_0), a\|] \end{aligned}$$

which implies that

$$(1 - \beta - 2\gamma) \|T^m(x_0) - T^n(x_0), a\| \leq (\alpha + \beta + \gamma) [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^{n-1}(x_0) - T^n(x_0), a\|]$$

gives

$$\|T^m(x_0) - T^n(x_0), a\| \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - 2\gamma} \right) [\|T^{m-1}(x_0) - T^m(x_0), a\| + \|T^{n-1}(x_0) - T^n(x_0), a\|]$$

where $\max\{\alpha, \beta\} + \gamma < \frac{1}{2}$.

Which tends to 0 as $m, n \rightarrow \infty$, since T is asymptotically regular in X . Then $\{T^n(x_0)\}$ is a Cauchy sequence. Since X is a 2-Banach space, $\lim_n T^n(x_0) = u \in X$. Then

$$\begin{aligned} \|u - T(u), a\| &\leq \|u - T^n(x_0), a\| + \|T^n(x_0) - T(u), a\| \\ &\leq \|u - T^n(x_0), a\| + \alpha [\|T^{n-1}(x_0) - T^n(x_0), a\| \\ &\quad + \|u - T(u), a\|] + \beta \|T^{n-1}(x_0) - u, a\| \\ &\quad + \gamma \max\{\|T^{n-1}(x_0) - T(u), a\|, \|u - T^n(x_0), a\|\} \end{aligned}$$

Letting $n \rightarrow \infty$, we get $\|u - T(u), a\| \leq (\alpha + \gamma) \|u - T(u), a\|$ implies $u = T(u)$. For uniqueness of u , let $u \neq v$ with $T(v) = v$ for $v \in X$. Then

$$\begin{aligned} \|u - v, a\| &= \|T(u) - T(v), a\| \\ &\leq \alpha [\|u - T(u), a\| + \|v - T(v), a\|] + \beta \|u - v, a\| \\ &\quad + \gamma \max\{\|u - T(v), a\|, \|v - T(u), a\|\} \end{aligned}$$

which implies $\|u - v, a\| \leq (\beta + \gamma) \|u - v, a\|$ gives a contradiction. Hence $u = v$. \square

Theorem 11. *Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and T be a mapping from X into itself satisfying (3.1). If T is asymptotically regular at some point $x \in X$ and the sequence of iterates $\{T^n(x)\}$ has a subsequence converging to a point $z \in X$, then z is the unique fixed point of T and $\{T^n(x)\}$ also converges to z .*

Proof. Let $\lim_k T^{n_k}(x) = z$. Then

$$\begin{aligned} \|z - T(z), a\| &\leq \|z - T^{n_k+1}(x), a\| + \|T^{n_k+1}(x) - T(z), a\| \\ &\leq \|z - T^{n_k+1}(x), a\| + \alpha [\|T^{n_k}(x) - T^{n_k+1}(x), a\| \\ &\quad + \|z - T(z), a\|] + \beta \|T^{n_k}(x) - z, a\| \\ &\quad + \gamma \max\{\|T^{n_k}(x) - T(z), a\|, \|z - T^{n_k+1}(x), a\|\} \end{aligned}$$

Letting $k \rightarrow \infty$ we get $\|z - T(z), a\| \leq (\alpha + \gamma) \|z - T(z), a\| \Rightarrow T(z) = z$. Also uniqueness of z follows very immediate.

$$\begin{aligned}
 \text{Next } \|z - T^n(x), a\| &= \|T(z) - T^n(x), a\| \\
 &\leq \alpha [\|z - T(z), a\| + \|T^{n-1}(x) - T^n(x), a\|] \\
 &\quad + \beta \|z - T^{n-1}(x), a\| + \gamma \max \{\|z - T^n(x), a\|, \\
 &\quad \|T^{n-1}(x) - T(z), a\|\} \\
 &\leq \alpha [\|z - T(z), a\| + \|T^{n-1}(x) - T^n(x), a\|] \\
 &\quad + \beta \|z - T^{n-1}(x), a\| + \gamma [\|z - T^n(x), a\| \\
 &\quad + \|T^{n-1}(x) - T(z), a\|] \\
 &\leq \alpha [\|z - T(z), a\| + \|T^{n-1}(x) - T^n(x), a\|] \\
 &\quad + \beta \|z - T^n(x), a\| + \beta \|T^n(x) - T^{n-1}(x), a\| \\
 &\quad + \gamma \|z - T^n(x), a\| + \gamma \|T^{n-1}(x) - T^n(x), a\| \\
 &\quad + \gamma \|T^n(x) - T(z), a\| \\
 &= \alpha \|z - T(z), a\| + (\alpha + \beta + \gamma) \|T^{n-1}(x) - T^n(x), a\| \\
 &\quad + (\beta + 2\gamma) \|z - T^n(x), a\|
 \end{aligned}$$

implies

$$(1 - \beta - 2\gamma) \|z - T^n(x), a\| \leq (\alpha + \beta + \gamma) \|T^{n-1}(x) - T^n(x), a\|$$

gives

$$\|z - T^n(x), a\| \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - 2\gamma} \right) \|T^n(x) - T^{n-1}(x), a\|$$

which tends to 0 as $n \rightarrow \infty$, since T is asymptotically regular in X . Thus $\lim_{n \rightarrow \infty} T^n(x) = z$. \square

Theorem 12. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and $\{T_n\}$ is a sequence of mappings from X into itself satisfying (3.1) with same constants α, β, γ and possessing fixed points u_n ($n = 1, 2, \dots$). Suppose that $T(x) = \lim_{n \rightarrow \infty} T_n(x)$ for $x \in X$. Then T has a unique fixed point u if and only if $u = \lim_n u_n$.

Proof. The proof is similar to that of Theorem 10 or Theorem 11. So we omit the proof here. \square

Another analogous theorem can similarly be proved.

Theorem 13. Let (X, d) be a 2-Banach space and $\{T_j\}$ be a sequence of mapping of X into itself satisfying

$$\begin{aligned}
 \|T_j(x) - T_j(y), a\| &\leq \alpha [\|x - T_j(x), a\| + \|y - T_j(y), a\|] + \beta \|x - y, a\| \\
 &\quad + \gamma \max \{\|x - T_j(y), a\|, \|y - T_j(x), a\|\} \quad (3.2)
 \end{aligned}$$

for every $x, y, a \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha, \beta\} + \gamma < \frac{1}{2}$. Suppose $T^n(x) = \lim_{j \rightarrow \infty} T_j^n(x)$ for all $x \in X$. Then T has a unique fixed point in X if T is asymptotically regular (a.r) at some point $x \in X$.

Proof. Let $T^n(x_0) = \lim_{j \rightarrow \infty} T_j^n(x_0)$ for $x_0 \in X$ Then for positive integers m, n

$$\begin{aligned} \|T_j^m(x_0) - T_j^n(x_0), a\| &= \left\| T_j(T_j^{m-1}(x_0)) - T_j(T_j^{n-1}(x_0)), a \right\| \\ &\leq \alpha \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| + \left\| T_j^{n-1}(x_0) - T_j^n(x_0), a \right\| \right] \\ &\quad + \beta \left\| T_j^{m-1}(x_0) - T_j^{n-1}(x_0), a \right\| \\ &\quad + \gamma \max \left\{ \left\| T_j^{m-1}(x_0) - T_j^n(x_0), a \right\|, \right. \\ &\quad \left. \left\| T_j^{n-1}(x_0) - T_j^m(x_0), a \right\| \right\} \\ &= \alpha \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| + \left\| T_j^{n-1}(x_0) - T_j^n(x_0), a \right\| \right] \\ &\quad + \beta \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| \right. \\ &\quad \left. + \left\| T_j^n(x_0) - T_j^{n-1}(x_0), a \right\| \right] \\ &\quad + \gamma \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| + \left\| T_j^m(x_0) - T_j^n(x_0), a \right\| \right. \\ &\quad \left. + \left\| T_j^{n-1}(x_0) - T_j^n(x_0), a \right\| + \left\| T_j^n(x_0) - T_j^m(x_0), a \right\| \right] \end{aligned}$$

which implies that

$$(1 - \beta - 2\gamma) \left\| T_j^m(x_0) - T_j^n(x_0), a \right\| \leq (\alpha + \beta + \gamma) \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| + \left\| T_j^{n-1}(x_0) - T_j^n(x_0), a \right\| \right]$$

gives

$$\begin{aligned} \left\| T_j^m(x_0) - T_j^n(x_0), a \right\| &\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - 2\gamma} \right) \left[\left\| T_j^{m-1}(x_0) - T_j^m(x_0), a \right\| \right. \\ &\quad \left. + \left\| T_j^{n-1}(x_0) - T_j^n(x_0), a \right\| \right] \end{aligned}$$

where $\max\{\alpha, \beta\} + \gamma < \frac{1}{2}$.

Letting $j \rightarrow \infty$ we get

$$\begin{aligned} \left\| T^m(x_0) - T^n(x_0), a \right\| &\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - 2\gamma} \right) \left[\left\| T^{m-1}(x_0) - T^m(x_0), a \right\| \right. \\ &\quad \left. + \left\| T^{n-1}(x_0) - T^n(x_0), a \right\| \right] \end{aligned}$$

where $\max\{\alpha, \beta\} + \gamma < \frac{1}{2}$.

Let T be asymptotically regular at some point $x_0 \in X$. Then right hand side of the inequality tends to 0 as $m, n \rightarrow \infty$ and hence $\{T^n(x_0)\}$ is a Cauchy sequence. Then by completeness of X , $\lim_n T^n(x_0) = u \in X$. Then

$$\begin{aligned} \|u - T(u), a\| &\leq \|u - T^n(x_0), a\| + \|T^n(x_0) - T^{n+1}(x_0), a\| \\ &\quad + \|T^{n+1}(x_0) - T(u), a\| \end{aligned} \quad (3.3)$$

$$\begin{aligned} \text{Now } \|T^{n+1}(x_0) - T(u), a\| &\leq \alpha [\|T^n(x_0) - T^{n+1}(x_0), a\| \\ &\quad + \|u - T(u), a\|] + \beta \|T^n(x_0) - u, a\| \\ &\quad + \gamma \max \{ \|T^n(x_0) - T(u), a\|, \\ &\quad \|u - T^{n+1}(x_0), a\| \} \end{aligned} \quad (3.4)$$

Then from (3.3) and (3.4) we get

$$\begin{aligned} \|u - T(u), a\| &\leq \|u - T^n(x_0), a\| + \|T^{n+1}(x_0) - T(u), a\| \\ &\quad + \alpha [\|T^n(x_0) - T^{n+1}(x_0), a\| \\ &\quad + \|u - T(u), a\|] + \beta \|T^n(x_0) - u, a\| \\ &\quad + \gamma \max \{ \|T^n(x_0) - T(u), a\|, \\ &\quad \|u - T^{n+1}(x_0), a\| \} \end{aligned} \quad (3.5)$$

Taking limit of (3.5) as $n \rightarrow \infty$ we get

$$\|u - T(u), a\| \leq (1 + \alpha + \gamma) \|u - T(u), a\|$$

and thus we obtain $u = T(u)$. The proof of uniqueness of u is similar to that proved in Theorem 10. \square

Here we state an open problem:

If all the respective conditions of Theorem 10 and Theorem 13 are satisfied for a map T having unique fixed point. Then do they force the map T to have asymptotic regularity property in $(X, \|\cdot, \cdot\|)$ in respective cases. We will consider this problem in a subsequent paper.

References

- [1] M.Açıkgoz, *A review on 2-normed structures*, Int. Journal of Math. Analysis, **1**(2007), no.4, 187 - 191. [MR2340939](#) (2008e:46012) [Zbl 1132.46304](#).
- [2] Y.J.Cho, M.S.Khan and S.L.Sing, *Common fixed points of weakly commuting mappings*, Univ.u. Novom Sadu, Zb.Rad. Period.-Mat.Fak.Ser.Mat, **18** 1(1988)129-142. [MR1034710](#)

Surveys in Mathematics and its Applications **7** (2012), 31 – 38

<http://www.utgjiu.ro/math/sma>

- [3] S.Gähler, *2-metric Raume and ihre topologische strucktur*, Math.Nachr., **26**(1963), 115-148. [MR0162224](#)
- [4] S.Gähler, *Uber die unifromisieberkeit 2-metrischer Raume*, Math.Nachr. **28**(1965), 235 - 244. [MR0178452](#)
- [5] K.Iseki, *Fixed point theorems in 2-metric space*, Math.Seminar.Notes, Kobe Univ.,**3**(1975), 133 - 136. [MR0405395](#)
- [6] C.Panja and A.P.Baisnab , *Asymptotic regularity and fixed point theorems*, The Mathematics Student, **46** 1(1978), 54-59 . [MR0698179](#)
- [7] B.E.Rhoades, *Contractive type mappings on a 2-metric space*, Math.Nachr., **91**(1979), 151 - 155. [MR0563606](#)
- [8] A.White, *2-Banach spaces*, Math.Nachr., **42**(1969), 43 - 60. [MR0257716](#) (41 # 2365) [Zbl 0185.20003](#)

Mantu Saha
 Department of Mathematics,
 The University of Burdwan,
 Burdwan 713104, West Bengal, India.
 e-mail: mantusaha@yahoo.com

Debashis Dey
 Koshigram Union Institution,
 Koshigram - 713150, Burdwan,
 West Bengal, India.
 e-mail: debashisdey@yahoo.com

Anamika Ganguly
 Balgona Saradmoni Balika Vidyalaya,
 Balgona Station and Chati, Bhatar,
 Burdwan, West Bengal, India.
 e-mail: anamika.simi@gmail.com

Lokenath Debnath
 Department of Mathematics,
 University of Texas - Pan American,
 Edinburg, Texas 78539, U.S.A.
 e-mail: debnathl@utpa.edu
