Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume **5** (2010), 285 – 295

A CLASS OF ANALYTIC FUNCTIONS BASED ON CONVOLUTION

K.G. Subramanian, T.V. Sudharsan, R. Thirumalaisamy and H. Silverman

Abstract. We introduce a class $TS_p^g(\alpha)$ of analytic functions with negative coefficients defined by convolution with a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n > 0, |z| < 1$. We obtain the coefficient inequality, coefficient estimate, distortion theorem, a convolution result, extreme points and integral representation for functions in the class $TS_p^g(\alpha)$.

1 Introduction

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. Let $S^*(\alpha)$ denote the subfamily of S consisting of functions starlike of order α, $0 \leq \alpha < 1$.

Let T be the class of all analytic functions with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, z \in \Delta) \tag{1.2}$$

The subclass of T consisting of starlike functions of order α denoted by $T^*(\alpha) = T \cap S^*(\alpha)$ was studied by Silverman [7]. In [8], the subclass $TS_p(\alpha)$ of functions of the form (1.2) for which

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad -1 \leq \alpha < 1$$

has been studied. We need the following theorem proved in [8].

Theorem 1. *A necessary and sufficient condition for f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ to be in $TS_p(\alpha)$, $-1 \leq \alpha < 1$ is that $\sum_{n=2}^{\infty}[2n - (\alpha+1)]a_n \leq 1 - \alpha$.*

2010 Mathematics Subject Classification: 30C45; 30C50.
Keywords: Analytic functions; Starlike functions; Convolution.

In [1], Ahuja has studied the class $T_\lambda(\alpha)$, consisting of functions $f(z) \in T$ satisfying the condition $Re\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} > \alpha$, $z \in \Delta, \lambda > -1, \alpha < 1$, where the operator $D^\lambda f$ is the Ruscheweyh derivative [5] of f defined by $D^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}$. In fact we can write

$$T_\lambda(\alpha) = \left\{ f \in T : Re\frac{z(f*g)'(z)}{(f*g)(z)} > \alpha, \quad g(z) = \frac{z}{(1-z)^{\lambda+1}} \right\},$$

where $f * g$ is the Hadamard product of analytic functions.

Recall that for any two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g*f)(z).$$

Classes of analytic functions defined by convolution with given analytic functions have been investigated for their properties (see for example [2, 3, 4, 9] to mention a few). In this paper, motivated by [1], we introduce and study a class $TS_p^g(\alpha)$, $-1 \le \alpha < 1$ where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n > 0, z \in \Delta$ is a fixed analytic function.

Definition 2. *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a fixed analytic function in $\Delta = \{z : |z| < 1\}$ and for $n \ge 2, b_n > 0$,. The class $TS_p^g(\alpha)$ consists of analytic functions of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where $z \in \Delta$ and for $n \ge 2, a_n \ge 0$, satisfying the inequality,*

$$Re\left\{ \frac{z(f*g)'(z)}{(f*g)(z)} - \alpha \right\} \ge \left| \frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right|, \quad -1 \le \alpha < 1$$

We obtain coefficient inequality, coefficient estimate, distortion theorem, convolution result, extreme points and integral representation for functions in the class $TS_p^g(\alpha)$. Unless mentioned otherwise, the function g is taken as in definition 2.

2 The class $TS_p^g(\alpha)$

We begin with a necessary and sufficient condition to be in the class $TS_p^g(\alpha)$.

Theorem 3. *A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in TS_p^g(\alpha)$ if and only if $\sum_{n=2}^{\infty} [2n - (\alpha+1)] a_n b_n \le 1 - \alpha$.*

Proof. From the definitions of the classes $TS_p^g(\alpha)$ and $TS_p(\alpha)$ it follows on using Theorem 1 that

$$f \in TS_p^g(\alpha) \Leftrightarrow f * g \in TS_p(\alpha) \Leftrightarrow \sum_{n=2}^{\infty}[2n-(\alpha+1)]a_n b_n \leq 1-\alpha.$$

□

Remark 4. *For* $-1 \leq \alpha < 1$ *and* $g(z) = \frac{z}{1-z}$*, we have* $TS_p^g(\alpha) = TS_p(\alpha)$ [8]

Theorem 5. *If* $f \in TS_p^g(\alpha)$*, then*

$$a_n \leq \frac{1-\alpha}{[2n-(\alpha+1)]b_n}$$

with equality only for the functions of the form

$$f_n(z) = z - \frac{1-\alpha}{[2n-(\alpha+1)]b_n} z^n.$$

Proof. If $f \in TS_p^g(\alpha)$, then

$$\begin{aligned} [2n-(\alpha+1)]a_n b_n &\leq \sum_{n=2}^{\infty}[2n-(\alpha+1)]a_n b_n \\ &\leq 1-\alpha \end{aligned}$$

implies $a_n \leq \frac{1-\alpha}{[2n-(\alpha+1)]b_n}$.

We have equality for

$$f_n(z) = z - \frac{(1-\alpha)z^n}{[2n-(\alpha+1)]b_n} \in TS_p^g(\alpha).$$

□

Theorem 6. *If* $f \in TS_p^g(\alpha)$*, then*

$$r - \frac{1-\alpha}{min[(2n-1-\alpha)b_n]} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{min[(2n-1-\alpha)b_n]} r^2,$$

$|z| = r < 1$. *The result is sharp for* $f(z) = z - \frac{1-\alpha}{min[(2n-1-\alpha)b_n]} z^2$.

Proof. Let $|z| = r < 1$. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^2 \\ &= r + r^2 \sum_{n=2}^{\infty} a_n \end{aligned} \tag{2.1}$$

Since $min[(2n-1-\alpha)b_n] \leq [2n-(\alpha+1)]b_n$, in view of Theorem 3,

$$\begin{aligned} min[(2n-1-\alpha)b_n] \sum_{n=2}^{\infty} a_n &\leq \sum_{n=2}^{\infty} [2n-(\alpha+1)]a_n b_n \\ &\leq 1-\alpha \end{aligned}$$

Hence

$$|f(z)| \leq r + r^2 \frac{1-\alpha}{min[(2n-1-\alpha)b_n]}.$$

Similarly we can show that $|f(z)| \geq r - r^2 \frac{1-\alpha}{min[(2n-1-\alpha)b_n]}$. □

We now prove a convolution result for the family $TS_p^g(\alpha)$.

Definition 7. *Let*

$$f_1(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$$

and

$$f_2(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n > 0.$$

Then the Quasi Hadamard product $(f_1 * f_2)(z)$ *is defined by*

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

Theorem 8. *Let $f \in TS_p^{k_1}(\alpha)$, $g \in TS_p^{k_2}(\alpha)$ where, for $z \in \Delta$,*

$$\begin{aligned} k_i(z) &= z + \sum_{n=2}^{\infty} b_{in} z^n, \quad b_{in} > 0 \text{ for } i = 1, 2 \\ h(z) &= z + \sum_{n=2}^{\infty} h_n z^n, \quad h_n > 0 \\ f(z) &= z - \sum_{n=2}^{\infty} f_n z^n, \quad f_n > 0 \\ g(z) &= z - \sum_{n=2}^{\infty} g_n z^n, \quad g_n > 0 \end{aligned}$$

then

$$f * g \in TS_p^h(\gamma), \ \ \gamma = \min_n G(n)$$

where

$$G(n) = \frac{[2n - (\alpha+1)][2n - (\beta+1)]b_{1n}b_{2n} - (2n-1)h_n(1-\alpha)(1-\beta)}{[2n - (\alpha+1)][2n - (\beta+1)]b_{1n}b_{2n} - h_n(1-\alpha)(1-\beta)}$$

provided $b_{1n}b_{2n} > (2n-1)h_n$.

Proof. Since $f \in TS_p^{k_1}(\alpha)$, we have

$$\sum_{n=2}^{\infty} \frac{[2n - (\alpha+1)]}{1-\alpha} f_n b_{1n} \leq 1 \tag{2.2}$$

Also $g \in TS_p^{k_2}(\beta)$ implies

$$\sum_{n=2}^{\infty} \frac{[2n - (\beta+1)]}{1-\beta} g_n b_{2n} \leq 1 \tag{2.3}$$

Now, we will show that

$$f * g = z - \sum_{n=2}^{\infty} f_n g_n z^n \in TS_p^h(\gamma) \text{ i.e. } \sum_{n=2}^{\infty} \frac{[2n - (\gamma+1)]}{1-\gamma} f_n g_n h_n \leq 1. \tag{2.4}$$

In order to prove that (2.2) and (2.3) imply (2.4), we note that

$$\begin{aligned} &\sum_{n=2}^{\infty} \left[\frac{[2n - (\alpha+1)][2n - (\beta+1)]}{(1-\alpha)(1-\beta)} f_n g_n b_{1n} b_{2n} \right]^{1/2} \\ \leq & \left(\sum_{n=2}^{\infty} \frac{[2n - (\alpha+1)]}{1-\alpha} f_n b_{1n} \right)^{1/2} \left(\sum_{n=2}^{\infty} \frac{[2n - (\beta+1)]}{1-\beta} g_n b_{2n} \right)^{1/2} \\ \leq & \ 1 \end{aligned} \tag{2.5}$$

In order to prove (2.4), it suffices to show that

$$\frac{[2n-(\gamma+1)]}{1-\gamma} f_n g_n h_n \leq \left[\frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} f_n g_n b_{1n} b_{2n}\right]^{1/2}$$

or equivalently

$$\sqrt{f_n g_n} \leq \frac{1-\gamma}{[2n-(\gamma+1)]h_n} \left[\frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n}\right]^{1/2} \tag{2.6}$$

From (2.5), we have

$$\sqrt{f_n g_n} \leq \sqrt{\frac{(1-\alpha)(1-\beta)}{[2n-(\alpha+1)][2n-(\beta+1)]b_{1n} b_{2n}}} \tag{2.7}$$

In view of (2.6) and (2.7), it is enough to prove that

$$\begin{aligned}
&\sqrt{\frac{(1-\alpha)(1-\beta)}{[2n-(\alpha+1)][2n-(\beta+1)]b_{1n} b_{2n}}} \\
&\qquad \leq \frac{1-\gamma}{[2n-(\gamma+1)]h_n} \left[\frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n}\right]^{1/2} \\
\Leftrightarrow \quad & \frac{2n-(\gamma+1)}{1-\gamma} \leq \frac{1}{h_n} \sqrt{\frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n}} \\
&\qquad \times \sqrt{\frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n}} \\
\Leftrightarrow \quad & \frac{2n-(\gamma+1)}{1-\gamma} \leq \frac{1}{h_n} \frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n} \\
&\qquad \Leftrightarrow \; 2n-(\gamma+1) \leq B - B\gamma
\end{aligned} \tag{2.8}$$

where

$$B = \frac{1}{h_n} \frac{[2n-(\alpha+1)][2n-(\beta+1)]}{(1-\alpha)(1-\beta)} b_{1n} b_{2n}.$$

Then $B \geq 2n-1 > 1$ and (2.8) is equivalent to

$$\gamma \leq \frac{B-(2n-1)}{B-1} \tag{2.9}$$

$$= \frac{1-\frac{2n-1}{B}}{1-\frac{1}{B}} \tag{2.10}$$

$$= \frac{1-\frac{(2n-1)h_n(1-\alpha)(1-\beta)}{[2n-(\alpha+1)][2n-(\beta+1)]b_{1n}b_{2n}}}{1-\frac{h_n(1-\alpha)(1-\beta)}{[2n-(\alpha+1)][2n-(\beta+1)]b_{1n}b_{2n}}} = G(n). \tag{2.11}$$

This proves the result. □

Theorem 9. *If*

$$f_1(z) = z - \sum_{n=2}^{\infty} a_n z^n, f_2(z) = z - \sum_{n=2}^{\infty} c_n z^n$$

and

$$f_3(z) = z - \sum_{n=2}^{\infty} (a_n^2 + c_n^2) z^n$$

where $f_1(z), f_2(z) \in TS_p^g(\alpha)$, *then* $f_3 \in TS_p^{g_1}(\beta)$ *where*

$$\beta = \min_n \left[\frac{[2n-(\alpha+1)]^2 - 2(2n-1)(1-\alpha)^2}{[2n-(\alpha+1)]^2 - 2(1-\alpha)^2} \right].$$

and $g_1(z) = (g * g)(z) = z + \sum_{n=2}^{\infty} b_n^2 z^n$.

Proof. Since $f_1(z) \in TS_p^g(\alpha)$, we have

$$\sum_{n=2}^{\infty} \frac{2n-(\alpha+1)}{1-\alpha} a_n b_n \leq 1$$

and hence

$$\sum_{n=2}^{\infty} \left(\frac{2n-(\alpha+1)}{1-\alpha} \right)^2 a_n^2 b_n^2 \leq 1.$$

Also since $f_2(z) \in TS_p^g(\alpha)$, we have

$$\sum_{n=2}^{\infty} \frac{2n-(\alpha+1)}{1-\alpha} c_n b_n \leq 1$$

and hence

$$\sum_{n=2}^{\infty} \left(\frac{2n-(\alpha+1)}{1-\alpha} \right)^2 c_n^2 b_n^2 \leq 1.$$

Therefore

$$\frac{1}{2} \sum_{n=2}^{\infty} \left[\frac{2n-(\alpha+1)}{1-\alpha} \right]^2 (a_n^2 + c_n^2) b_n^2 \leq 1.$$

In view of Theorem 3, we have to show that

$$\sum_{n=2}^{\infty} \left[\frac{2n-(\beta+1)}{1-\beta} \right] (a_n^2 + c_n^2) b_n^2 \leq 1$$

The last inequality will be satisfied if

$$\frac{2n-(\beta+1)}{1-\beta} \leq \frac{1}{2} \left[\frac{2n-(\alpha+1)}{1-\alpha} \right]^2.$$

Solving for β, we have

$$\beta \leq \frac{[2n-(\alpha+1)]^2 - 2(2n-1)(1-\alpha)^2}{[2n-(\alpha+1)]^2 - 2(1-\alpha)^2}.$$

Hence the result follows. □

3 Extreme points and Integral representation

Theorem 10. *Let*

$$f_1(z) = z, f_n(z) = z - \frac{1-\alpha}{[2n-(\alpha+1)]b_n} z^n,$$

$n = 2, 3, \ldots$ *then* $f \in TS_p^g(\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z);$$

where $\mu_n \geq 0$ *and* $\sum_{n=1}^{\infty} \mu_n = 1$. *In particular the extreme points of* $TS_p^g(\alpha)$ *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\alpha}{[2n-(\alpha+1)]b_n} z^n,$$

$n = 2, 3, \ldots.$

Proof. First let f be expressed as in the above theorem. This means that we can write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(1-\alpha)\mu_n}{[2n-(\alpha+1)]b_n} z^n \\ &= z - \sum_{n=2}^{\infty} t_n z^n \end{aligned}$$

Therefore $f \in TS_p^g(\alpha)$, since $\sum_{n=2}^{\infty} \frac{2n-(\alpha+1)}{1-\alpha} t_n b_n = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 < 1$

Conversely, If $f \in TS_p^g(\alpha)$, by Theorem 3, we have

$$a_n \leq \frac{1-\alpha}{[2n-(\alpha+1)]b_n}$$

so we may set $\mu_n = \frac{[2n-(\alpha+1)]a_n b_n}{1-\alpha}$ and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$.

Then

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[2n-(\alpha+1)]b_n} \mu_n z^n \\ &= z - \sum_{n=2}^{\infty} \mu_n [z - f_n(z)] \\ &= \left(1 - \sum_{n=2}^{\infty} \mu_n\right) z + \sum_{n=2}^{\infty} \mu_n f_n(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \end{aligned}$$

□

Remark 11.

1. *For* $g(z) = \frac{z}{1-z}$*, we obtain corollary 1 in [8].*
2. *For* $g(z) = \frac{z}{1-z}$ *and replacing* α *by* $\frac{1+\alpha}{2}$*, in* $T^*(\alpha)$*, we obtain the corresponding result in [7].*

Theorem 12. *If* f *is in* $TS_p^g(\alpha)$*, then*

$$(f * g)(z) = exp \int_0^z \frac{1+\alpha\rho(t)}{t(1-\rho(t))} dt$$

for some $\rho(z)$*,* $|\rho(z)| < 1, z \in \Delta$ *where*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

$b_n > 0$ *for* $n = 2, 3, \ldots$ *Also*

$$(f * g)(z) = z \ \exp\left[\int_x ln(1-xz)^{-(1+\alpha)} d\mu(x)\right]$$

where $\mu(x)$ *is probability measure on* $X = \{x : |x| = 1\}$*.*

Proof. Let $f \in TS_p^g(\alpha)$ and

$$\omega = \frac{z(f*g)'(z)}{(f*g)(z)}.$$

We have $Re\ \omega - \alpha > |\omega - 1|$. Therefore

$$\left|\frac{\omega - 1}{\omega - \alpha}\right| < 1$$

and

$$\frac{\omega-1}{\omega-\alpha}=\rho(z),$$

where $|\rho(z)|<1, z\in\Delta$.
This gives

$$\frac{(f*g)'(z)}{(f*g)(z)}=\frac{1-\alpha\rho(z)}{z(1-\rho(z))}$$

and therefore

$$(f*g)(z)=exp\left[\int_0^z \frac{1-\alpha\rho(t)}{t(1-\rho(t))}dt\right].$$

Now set $X=\{x:|x|=1\}$. Then we have $\frac{\omega-1}{\omega-\alpha}=xz$ and hence

$$\frac{(f*g)'(z)}{(f*g)(z)} = \frac{1-\alpha xz}{z(1-xz)}=\frac{1}{z}+\frac{(1-\alpha)x}{1-xz}.$$

Integrating we obtain,

$$ln\frac{(f*g)(z)}{z} = (-1+\alpha)ln(1-xz).$$

This proves the second representation

$$(f*g)(z)=z\ \exp\left[\int_x ln(1-xz)^{-(1+\alpha)}d\mu(x)\right].$$

□

Remark 13. *Taking* $g(z)=\frac{z}{(1-z)^{\lambda+1}}$, $\lambda>-1$, $TS_p^g(\alpha)$ *reduces to the class* $D(\alpha,\beta,\lambda)$ *of [6], so that Theorems 2.1, 2.4, 2.7 in [6] are consequences of our Theorems 3, 10, 12.*

Acknowledgment. The authors thank the referees for very useful comments which helped to correct an error in Theorem 9 and to improve the paper.

References

[1] O.P. Ahuja, *Hadamard products of analytic functions defined by Ruscheweyh derivatives*, Current topics in Analytic Funtion Theory (H.M. Srivastava, S. Owa editors) World Scientific, Singapore 1992,
13-29. MR1232426(94j:30006). Zbl 1002.30007.

[2] K. Al-Shaqsi and M. Darus, *Fekete-Szegö Problem for Univalent Functions with Respect to k-Symmetric Points*, The Austral. J. Math. Anal. Appl., **5**(2) (2008) Art. No.6. MR2461677(2009m:30035). Zbl 1165.30308.

[3] M. Haji Mohd, R. M. Ali, S.K. Lee and V. Ravichandran, *Subclasses of Meromorphic Functions Associated with Convolution*, J. Ineq. Appl. Vol. **2009** (2009), Article ID 190291, 10 pages MR2496271(2010k:30009). Zbl 1176.30044.

[4] G. Murugusundaramoorthy and R.K. Raina, *On a certain class of analytic functions associated with a convolution structure*, Demonstratio Mathematica, **41**(3) (2008) 551-559. MR2433307(2009h:30022). Zbl 1176.30044.

[5] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109-115. MR0367176(51 #3418). Zbl 0303.30006.

[6] S. Shams, S.R. Kulkarni and J.M. Jahangiri, *On a class of univalent functions defined by Ruscheweyh derivatives*, Kyungpook Math. J. **43** (2003), 579-585. MR2026116(2004j:30026). Zbl 1067.30032.

[7] H. Silverman, *Univalent function with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), pp. 109-116. MR0369678(51#5910). Zbl 0311.30007.

[8] K.G. Subramanian, T.V. Sudharsan, P. Balasubrahmanyam and H. Silverman, *Classes of uniformly starlike functions*, Publ. Math, Debrecen **53**/3-4 (1998), 309-315. MRMR1657542(99i:30028). Zbl 0921.30007.

[9] S. Sivaprasad Kumar , V. Ravichandran and H. C. Taneja, *Meromorphic Functions with positive coefficients defined using convolution*, J. Ineq. Pure and Appl. Math., **6**(2) (2005), Article No. 58. MR2150912. Zbl 1080.30012.

K.G. Subramanian
School of Mathematical sciences,
Universiti Sains Malaysia,
11800 Penang, Malaysia.
e-mail: kgsmani1948@yahoo.com

T.V. Sudharsan
Department of Mathematics,
SIVET College,
Chennai 601 302 India.
e-mail: tvsudharsan@rediffmail.com

R. Thirumalaisamy
Department of Mathematics,
Government Arts College,
Chennai 600 035 India.
e-mail: raathi912@yahoo.com

H. Silverman
Department of Mathematics,
College of Charleston,
Charleston SC 29424, USA.
e-mail: silvermanh@cofc.edu