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EXISTENCE OF POSITIVE SOLUTION TO A QUASILINEAR ELLIPTIC PROBLEM IN R^N

Dragoş-Pătru Covei

Abstract. In this paper we prove the existence of positive solution for the following quasilinear problem

$$\begin{aligned} -\Delta_p u &= a(x)f(u), \text{ in } \mathbb{R}^N, \\ u &> l > 0, \text{ in } \mathbb{R}^N, \\ u(x) &\to l, \text{ as } |x| \to \infty, \end{aligned}$$

where $\Delta_p u$, $(1 < p < \infty)$ is the p-Laplacian operator. The proof is based on the results due to Diaz-Saà ([2]).

1 Introduction

Let us consider the problem

$$\begin{gathered} -\Delta_p u = a(x)f(u), \text{ in } \mathbb{R}^N, \\ u > l > 0, \text{ in } \mathbb{R}^N, \\ u(x) \to l, \text{ as } |x| \to \infty, \end{gathered} \tag{1}$$

where $N > 2$, $\Delta_p u$, $(1 < p < \infty)$ is the p-Laplacian operator, $l > 0$ is a real number and the function $a(x)$ satisfies the following hypotheses:

(A1) $a(x) \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$;

(A2) $a(x) > 0$ in $\mathbb{R}^N$;

(A3) For $\Phi(r) = \max_{|x|=r} a(x)$ and $p < N$,

$$0 < \int_1^\infty r^{1/(p-1)} \Phi^{1/(p-1)}(r) dr < \infty \quad \text{if } 1 < p \leq 2$$

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$$0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} \Phi(r)dr < \infty \quad \text{if } 2 \le p < \infty\,.$$

and $f : (0,\infty) \to (0,\infty)$ be a C^1 function that satisfies the following assumptions:

(F1) $u \mapsto f(u)/u^{p-1}$ is decreasing on $(0,\infty)$;

(F2) $\lim_{u \searrow 0} \frac{f(u)}{u^{p-1}} = +\infty$.

Goncalves-Santos ([4]) solved the problem (1) in the case $l = 0$. In this article consider the problem when $l > 0$. The problem (1) arises, for example, in non-Newtonian fluid theory, the quantity p is a characteristic of the medium. The case $1 < p < 2$ corresponds to pseudoplastics fluids and $p > 2$ arises in the consideration of dilatant fluids.

Our main results are the following:

Theorem 1. *Under the hypotheses (F1), (F2), (A1)-(A3), problem (1) has a positive solution, $u \in C^{1,\alpha}(\mathbb{R}^N)$.*

2 Existence of a positive solution

To prove the existence of a solution of problem (1) we use an existence result of Diaz-Saà ([2, Theorem 1-2]). They considered the problem

$$\begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega \\ u \ge 0, \quad u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a open boundary regular and $g(x,u) : \Omega \times [0,\infty) \to \mathbb{R}$ satisfied the following hypotheses:

(H1) for a. e. $x \in \Omega$ the function $u \to g(x,u)$ is continuous on $[0,\infty)$ and the function $u \to g(x,u)/u^{p-1}$ is decreasing on $(0,\infty)$;

(H2) for each $u \ge 0$ the function $x \to g(x,u)$ belongs to $L^\infty(\Omega)$;

(H3) $\exists C > 0$ such that $g(x,u) \le C(u^{p-1} + 1)$ a.e. $x \in \Omega$, $\forall\, u \ge 0$.

Set

$$a_0(x) = \lim_{u \searrow 0} g(x,u)/u^{p-1} \qquad \text{and} \qquad a_\infty(x) = \lim_{u \to \infty} g(x,u)/u^{p-1}\,,$$

so that $-\infty < a_0(x) \le +\infty$ and $-\infty \le a_\infty(x) < +\infty$.

Under these hypotheses on g, Diaz-Saà ([2]) proved that there exist one solution of (2).

To prove the main Theorem we need the Diaz-Saà's inequality:

Lemma 2 ([2]). *For $i = 1,2$ let $w_i \in L^\infty(\Omega)$ such that $w_i > 0$ a.e. in Ω, $w_i \in W^{1,p}(\Omega)$, $\Delta_p w_i^{1/p} \in L^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. Then*

$$\int_\Omega \left(\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}}\right)(w_1 - w_2) \geq 0,$$

if $(w_i/w_j) \in L^\infty(\Omega)$ for $i \neq j, i, j = 1, 2$.

The first step, in the study of existence, is to observe that the problem (1) can be rewritten

$$\begin{cases} -\Delta_p v = a(x) f(v+l)\,, \text{in } \mathbb{R}^N, \\ v(x) > 0, \text{ in } \mathbb{R}^N, \\ v(x) \to 0\,, \quad \text{as} \quad |\, x \,| \to \infty\,. \end{cases} \tag{3}$$

To solve (3), for any positive integer k we consider the problem

$$\begin{cases} -\Delta_p v_k = a(x) f(v_k + l)\,, \text{ in } \quad B_k(0) \\ v_k(x) > 0, \text{ in } \quad B_k(0) \\ v_k(x) = 0\,, \quad \text{if} \quad |\, x \,| = k\,. \end{cases} \tag{4}$$

To obtain a solution to (4), it is sufficient to verify that the hypotheses of the Diaz-Saà theorem are fulfilled:

H1: since $f \in C^1((0,\infty),(0,\infty))$ and $l > 0$, it follows that the mapping $v \to a(x)f(v+l)$ is continuous in $[0,\infty)$ and from $a(x)\frac{f(v+l)}{v^{p-1}} = a(x)\frac{f(v+l)}{(v+l)^{p-1}} \cdot \frac{(v+l)^{p-1}}{v^{p-1}}$, using positivity of a and (F1) we deduce that the function $u \to a(x)\frac{f(v)}{v^{p-1}}$ is decreasing on $(0,\infty)$;

H2: for all $v \geq 0$, since $a(x) \in C^{0,\alpha}(\mathbb{R}^N)$, we obtain $x \to a(x)f(v)$ belongs to $L^\infty(\Omega)$;

H3: By $\lim_{v\to\infty} \frac{f(v+l)}{v^{p-1}+1} = \lim_{v\to\infty} \frac{f(v+l)}{(v+l)^{p-1}} \cdot \frac{(v+l)^{p-1}}{v^{p-1}+1} = 0$ and $f \in C^1$, there exists $C > 0$ such that $f(v+l) \leq C(v^{p-1}+1)$ for all $v \geq 0$. Therefore, $a(x)f(v+l) \leq \|a\|_{L^\infty(B_k(0))}\,(v^{p-1}+1)$ for all $v \geq 0$.

Observe that

$$a_0(x) = \lim_{v \searrow 0} \frac{a(x)f(v+l)}{v^{p-1}} = +\infty$$

and

$$a_\infty(x) = \lim_{v\to+\infty} \frac{a(x)f(v+l)}{v^{p-1}} = 0.$$

Thus by Diaz-Saa, problem (4) has a unique solution $v_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Applying the regularity theory ([3],[5],[6]) for elliptic equations we find $v_k \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$. Moreover, by the maximum principle, this solution is positive in $B_k(0)$.

In outside of $B_k(0)$ we define $v_k = 0$. We prove that $v_k \leq v_{k+1}$. Assume the contrary and let $w_1 := (v_k)^p$, $w_2 := (v_{k+1})^p$ in Diaz-Saà's inequality. Then

$$\begin{aligned} 0 &\leq \int_{\{x\in\mathbb{R}^N|v_k>v_{k+1}\}\subset B_k(0)} \left(\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}}\right)(w_1 - w_2) \\ &= \int_{\{x\in\mathbb{R}^N|v_k>v_{k+1}\}\subset B_k(0)} \left(\frac{-\Delta_p v_k}{v_k^{p-1}} + \frac{\Delta_p v_{k+1}}{v_2^{p-1}}\right)(v_k^p - v_{k+1}^p) \\ &= \int_{\{x\in\mathbb{R}^N|v_k>v_{k+1}\}\subset B_k(0)} a(x)\left(\frac{f(v_k+l)}{v_k^{p-1}} - \frac{f(v_{k+1}+l)}{v_{k+1}^{p-1}}\right)(v_k^p - v_{k+1}^p) < 0, \end{aligned}$$

which is impossible. Hence $v_k \leq v_{k+1}$. We now justify the existence of a continuous function $V : \mathbb{R}^N \to \mathbb{R}$ such that $v_k \leq V$ in $\mathbb{R}^N$. We first construct a positive radially symmetric function w such that $-\Delta_p w = \Phi(r)$, $(r = |x|)$ in $\mathbb{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. A straightforward computation shows that

$$w(r) := K - \int_0^r \left[\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\right]^{1/(p-1)} d\xi,$$

where

$$K = \int_0^\infty \left[\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\right]^{1/(p-1)} d\xi.$$

By result in ([4]) we remark that (A3) implies

$$\int_0^{+\infty} \left[\xi^{1-N} \int_0^\xi \sigma^{N-1}\Phi(\sigma)d\sigma\right]^{1/(p-1)} d\xi,$$

is finite.

An upper-solution to (1) will be constructed. Consider the function $\overline{f}(v) = (f(v+l)+1)^{1/(p-1)}$, for $u > 0$.

We have

(F1') $\overline{f}(v) \geq f(v+l)^{1/(p-1)}$

(F2') $\lim_{v\searrow 0} \overline{f}(v)/v = \infty$ and $v \mapsto \overline{f}(v)/v^{p-1}$ is decreasing on $(0,\infty)$.

Let V be a positive function such that $w(r) = \frac{1}{C}\int_0^{V(r)} t^{p-1}/\overline{f}(t)\,dt$, where C is a positive constant such that $KC \leq \int_0^{C^{1/(p-1)}} t^{p-1}/\overline{f}(t)\,dt$. In the case when $p = 2$, this method was introduced by Zhang in ([8]). We prove that we can find $C > 0$ with this property. From our hypothesis (F2') we obtain that $\lim_{x\to+\infty} \int_0^x t^{p-1}/\overline{f}(t)\,dt = +\infty$. Now using L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{1}{x^{p-1}} \int_0^x \frac{t^{p-1}}{\overline{f}(t)}dt = \lim_{x\to\infty} \frac{x}{(p-1)\overline{f}(x)} = +\infty.$$

This means that there exists $x_1 > 0$ such that $\int_0^x t^{p-1}/\overline{f}(t)\,dt \geq Kx^{p-1}$, for all $x \geq x_1$. It follows that for any $C \geq x_1$,

$$KC \leq \int_0^{C^{1/(p-1)}} \frac{t^{p-1}}{\overline{f}(t)} dt.$$

But w is a decreasing function, and this implies that V is a decreasing function too. Then

$$\int_0^{V(r)} \frac{t^{p-1}}{\overline{f}(t)} dt \leq \int_0^{V(0)} \frac{t^{p-1}}{\overline{f}(t)} dt = C \cdot w(0) = C \cdot K \leq \int_0^{C^{1/(p-1)}} \frac{t^{p-1}}{\overline{f}(t)} dt.$$

It follows that $V(r) \leq C^{1/(p-1)}$ for all $r > 0$. From $w(r) \to 0$ as $r \to +\infty$ we deduce $V(r) \to 0$ as $r \to +\infty$. By the choice of V we have

$$\Delta_p w = \frac{1}{C^{p-1}}(\frac{V^{p-1}}{\overline{f}(V)})^{p-1}\Delta_p V + (p-1)\frac{1}{C^{p-1}}|\nabla V|^p(\frac{V^{p-1}}{\overline{f}(V)})^{p-2}(\frac{V^{p-1}}{\overline{f}(V)})'. \tag{5}$$

From ((5)) and the fact that $v \to \frac{\overline{f}(v)}{v^{p-1}}$ is a decreasing function on $(0, +\infty)$, we deduce that

$$\Delta_p V \leq C^{p-1}(\frac{\overline{f}(V)}{V^{p-1}})^{p-1}\Delta_p w = -C^{p-1}(\frac{\overline{f}(V)}{V^{p-1}})^{p-1}\Phi(r) \leq -f(V)\Phi(r). \tag{6}$$

We prove that $v_k \leq V$. Assume the contrary and let $w_1 := (v_k)^p$, $w_2 := (V)^p$ in Diaz-Saà's inequality. Then

$$\begin{aligned} 0 &\leq \int_{\{x\in\mathbb{R}^N | v_k > V\}\subset B_k(0)} (\frac{-\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}})(w_1 - w_2) \\ &= \int_{\{x\in\mathbb{R}^N | v_k > V\}\subset B_k(0)} (\frac{-\Delta_p v_k}{v_k^{p-1}} - \frac{\Delta_p V}{V^{p-1}})(v_k^p - V^{p-1}) \\ &= \int_{\{x\in\mathbb{R}^N | v_k > V\}\subset B_k(0)} a(x)(\frac{f(v_k + l)}{v_k^{p-1}} - \frac{f(V + l)}{V^{p-1}})(v_k^p - V^p) < 0, \end{aligned}$$

which is impossible.

Hence $v_k \leq V$ for all $x \in \mathbb{R}^N$. It follows by the Diaz-Saà's inequality that

$$v_1 \leq v_2 \leq \cdots \leq v_k \leq \ldots \leq V, \text{ for all } x \in \mathbb{R}^N$$

with V vanishing at infinity. Thus there exists a function, say $v \leq V$ such that $v_k \to v$ pointwise in $\mathbb{R}^N$. Using the elliptic regularity theory ([3],[5],[6]) again we find that $v \in C^{1,\alpha}(\mathbb{R}^N)$. Then $u = v + l$ satisfies (1).

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University Constantin Brâncuşi of Târgu-Jiu,
Bld. Republicii 1, 210152, Târgu-Jiu,
Romania.
e-mail: dragoscovei@utgjiu.ro
http://www.utgjiu.ro/math/dcovei/