Componentwise conformal vector fields on Riemannian almost product manifolds

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Abstract. On a Riemannian almost product manifold, the notion of a componentwise conformal vector field is introduced and several examples are exhibited. We show that this class of vector fields is a conformal invariant. For a compact manifold, a Bochner type integral formula for the Ricci tensor on such vector fields is obtained. Then, integral inequalities which link a curvature condition with the existence of componentwise conformal vector fields are obtained. Also, applications to Riemaniann submersions are given, obtaining a new characterization of the standard flat n-torus.

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1 Introduction

K. Yano was the first to study systematically Riemannian almost product manifolds in a general setting [15]. A. Gray also worked with this notion, introducing the configuration tensors and derived several formulae which generalized classical ones of Riemann Geometry such as Gauss and Codazzi equations [7]. Essentially, a Riemannian almost product manifold is a Riemannian manifold (M, g) equipped with two complementary orthogonal distributions or, in a equivalent way, M is endowed with an isometric operator P satisfying $P^2 = \mathrm{Id}$. For instance, the total space of a Riemannian submersion admits such a structure. In this case, the vertical distribution is always integrable. Note that this is not the situation for a general Riemannian almost product manifold, where both distributions are interchangeable, in general. That is, a priori none of the two complementary orthogonal distributions satisfies any property that makes it special with respect to the other distribution. A general scheme for the classification of the Riemannian almost product manifolds was introduced by A. Naveira (cf. [10]), who considered the notions of anti-foliations, minimal or umbilical Riemannian almost product manifolds (see also [9] and references therein.)

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In this paper, we shall study a natural family of conformal-like (but not conformal, in general) vector fields on a compact Riemannian almost product manifold and its relation with curvature. Thus, we introduce the new notion of *componentwise conformal vector field* in Definition 2.1. Roughly speaking, such a vector field behaves as a conformal one when restricted to the (± 1)-eigenspaces of P, $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, but with (possibly) different conformal factors. In case these two conformal factors are equal, the usual notion of conformal vector field is included properly (see Example 4.2.) Indeed, several examples in Sections 2 and 4 show that this notion has a clear geometric meaning. In particular, on a Riemaniann submersion with totally umbilical fibers, the horizontal lift of a conformal vector field provides an example of our notion which is not necessarily a conformal vector field (Example 4.1.) The main aim of this paper is to obtain an integral formula which relates the existence of componentwise conformal vector fields and curvature properties of M when it is compact (Theorem 3.1), namely

> *For any componentwise conformal vector field K on a compact Riemannian almost product manifold M, it holds*
>
> $$\int_M \left\{ \mathrm{Ric}(K,K) + \frac{1}{2}\,\|\alpha_K\|^2 - \|\nabla K\|^2 + \Phi(\rho_1,\rho_2) \right\} \mathrm{d}\mu_g = 0,$$
>
> *where α_K is the symmetric tensor field introduced in Lemma* 2.1*, $\Phi(\rho_1,\rho_2) = n_1(2-n_1)\rho_1^2 + n_2(2-n_2)\rho_2^2 - 2n_1n_2\rho_1\rho_2$, $n_1 = \dim\mathcal{D}$, $n_2 = \dim\mathcal{D}^{\perp}$ and ρ_1, ρ_2 are the functions given in Definition* (2.1)*.*

In addition, we show obstruction results and further applications to the relevant case of Riemannian submersions.

The paper is organized as follows. In Section 2, we introduce the notion of componentwise conformal vector fields, expressing it in two more different equivalent ways (Lemmattas 2.1 and 2.2.) Moreover, several examples are exhibited in order to analyse basic properties of such vector fields. Componentwise conformal vector fields are conformal invariant (Example 2.5.) However, the set of all componentwise conformal vector fields is not a Lie algebra in general (Proposition 2.3, Remark 2.6 and Example 2.8.) Section 3 is devoted to the statement of the main result of this paper (Theorem 3.1.) Furthermore, several of its consequences are shown. The key tool of the proof is the classical Bochner's formula. Section 4 is devoted to particularizing our general integral formula to Riemannian submersions (Theorem 4.2.) We conclude this paper with some results inspired by the classical Bochner's technique. In this way, we obtain Theorem 4.4, which might be seen as a version for Riemannian submersions of a classical result by Bochner [2] (see also [13, Prop. 5.7]), namely,

> *Let $p : M \to B$ be a Riemannian submersion with totally umbilical fibers, where M is compact. Assume the Ricci tensor of M is negative semidefinite on horizontal vectors. Then, each Killing vector field on B must be parallel.*

Last, but not least, when the base manifold B has a *wide enough* isometry group, the previous result can be rewritten as follows (Corollary 4.5).

> *Let $p : M \to B$ be a Riemannian submersion with totally umbilical fibers, with M compact and B a Riemannian homogeneous manifold. Assume the*

Ricci tensor of M is negative semidefinite on horizontal vectors. Then, the horizontal distribution is integrable. Moreover, if the dimension of the fibers is greater or equal to 2, the fibers are totally geodesic and, up to a finite cover, B is isometric to a standard flat torus.

2 Concept and examples

Let (M, g) be a connected Riemannian manifold. An almost product structure on a manifold M is a tensor field $P \in \mathcal{T}_{(1,1)}(M)$ such that $P^2 = \mathrm{Id}$. The almost product structure P is called improper whenever $P = \pm Id$. Along this paper, any almost product will not be improper, unless otherwise stated.

We assume that there is an almost product structure P satisfying the condition $g(P(X), P(Y)) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. The triple (M, g, P) is called a Riemannian almost product manifold. We denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the orthogonal complementary distributions associated with the 1 and -1 eigenvalues of P, respectively. The corresponding projections π and $\pi^\perp$ onto $\mathcal{D}$ and $\mathcal{D}^\perp$ fulfil respectively

$$\pi = \frac{1}{2}(\mathrm{Id} + P), \quad \pi^\perp = \frac{1}{2}(\mathrm{Id} - P). \tag{2.1}$$

Conversely, assume two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are given on a Riemannian manifold (M, g). Then, we can easily define an almost product structure P such that (M, g, P) is a Riemannian almost product manifold.

Definition 2.1. A vector field $K \in \mathfrak{X}(M)$ is said to be componentwise conformal on (M, g, P) if there exist two (smooth) functions ρ_1, ρ_2 on M such that the Lie derivative of g respect to K, $\mathcal{L}_K g$, satisfies

1. $(\mathcal{L}_K g)(E, F) = 2\rho_1\, g(E, F)$ for any $E, F \in \mathcal{D}$, and
2. $(\mathcal{L}_K g)(E, F) = 2\rho_2\, g(E, F)$ for any $E, F \in \mathcal{D}^\perp$.

We will denote $n_1 = \dim \mathcal{D}$ and $n_2 = \dim \mathcal{D}^\perp$.

The following result shows an equivalent definition to the previous one.

Lemma 2.1. *A vector field $K \in \mathfrak{X}(M)$ is componentwise conformal on (M, g, P) if, and only if, there exist two (smooth) functions ρ_1, ρ_2 on M and a symmetric tensor field $\alpha_K \in \mathcal{T}_{(0,2)}(M)$ such that*

$$\mathcal{L}_K g = 2\rho_1\, g(\pi, \pi) + 2\rho_2\, g(\pi^\perp, \pi^\perp) + \alpha_K,$$

with $\alpha_K(\mathcal{D}, \mathcal{D}) = \alpha_K(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0$.

Proof. The sufficient condition is trivial, so we will focus on the necessary one. By using that for each $E \in \mathfrak{X}(M)$, $E = \pi(E) + \pi^\perp(E)$, we obtain

$$\begin{aligned}(\mathcal{L}_K g)(E, F) =& (\mathcal{L}_K g)(\pi(E), \pi(F)) + (\mathcal{L}_K g)(\pi(E), \pi^\perp(F)) \\ &+ (\mathcal{L}_K g)(\pi^\perp(E), \pi(F)) + (\mathcal{L}_K g)(\pi^\perp(E), \pi^\perp(F)) \\ =& 2\rho_1\, g(\pi(E), \pi(F)) + 2\rho_2\, g(\pi^\perp(E), \pi^\perp(F)) + \alpha_K(E, F),\end{aligned}$$

where

$$\alpha_K(E,F) = (\mathcal{L}_K g)(\pi(E), \pi^{\perp}(F)) + (\mathcal{L}_K g)(\pi^{\perp}(E), \pi(F)).$$

Clearly, α_K is symmetric. Finally, if $E, F \in \mathcal{D}$ or $E, F \in \mathcal{D}^{\perp}$, then it holds $\alpha_K(E,F) = 0$. □

Another equivalent notion to Definition 2.1 is given in the following result.

Lemma 2.2. *A vector field $K \in \mathfrak{X}(M)$ is componentwise conformal on (M, g, P) if, and only if, there exist two (smooth) functions λ, μ on M such that*

$$\mathcal{L}_K g = \lambda\, g + \mu\, \widehat{P} + \alpha_K, \tag{2.2}$$

where $\widehat{P}(E,F) = g(P(E),F)$, for $E, F \in \mathfrak{X}(M)$.

Proof. A straightforward computation from (2.1) and Lemma 2.1. Note that $\lambda = \rho_1 + \rho_2$ and $\mu = \rho_1 - \rho_2$. □

Remark 2.2. The tensor α_K satisfies $\alpha_K(V,X) = g([V,K],X) + g(V,[X,K])$ for $V \in \mathcal{D}$ and $X \in \mathcal{D}^{\perp}$. In the particular case that $\mathcal{D}^{\perp}$ is integrable and $K \in \mathcal{D}^{\perp}$, the above formula reduces to $\alpha_K(V,X) = g(X,[V,K])$. Note that similar computations can be done when $K \in \mathcal{D}$. Recall that the mean curvature vector field of an $(n_1 \geq 1)$-dimensional distribution $\mathcal{D}$ in a Riemannian manifold is given by

$$\mathbf{H} = \frac{1}{n_1} \sum_{i=1}^{n_1} \pi^{\perp}(\nabla_{V_j} V_j),$$

where $V_1, ..., V_{n_1}$ is a local orthonormal frame spanning $\mathcal{D}$. Assume $K \in \mathcal{D}^{\perp}$ is componentwise conformal, then

$$\rho_1 = -g(\mathbf{H}, K), \quad \text{and} \quad n_2\, \rho_2 = \mathrm{Tr}([\,-\,, K]|_{\mathcal{D}^{\perp}}).$$

Example 2.3. There are two trivial cases of Definition 2.1. The first one is when K is a conformal vector field of M, which obviously is componentwise conformal for $P = \mathrm{Id}$. The second one appears when (M,g) is a Riemaniann product $(M_1 \times M_2, g_1 + g_2)$ and $K = (K_1, K_2)$ where K_i is a conformal vector field on (M_i, g_i), $i = 1, 2$. Note that in both situations, the symmetric tensor fields α's vanish identically, which does not always hold.

Example 2.4. We recall that an *orthogonally conformal* vector field, [12], is a unit vector field Z on an $(n \geq 2)$-dimensional Riemannian manifold (M, g) such that for any $U, V \perp Z$, we have $(\mathcal{L}_Z g)(U,V) = 2\rho\, g(U,V)$, for a (smooth) function ρ on M. Consider the almost product structure P given by $P(Z) = Z$ and $P(X) = -X$ when $X \in Z^{\perp}$. We clearly have that Z is componentwise conformal. Indeed, we just take $\rho_1 = 0$, $\rho_2 = \rho$ and items 1 and 2 of Definition 2.1 are automatically satisfied.

Example 2.5. Let (M, g, P) be a Riemannian almost product manifold and assume it admits a componentwise vector field K. Also, consider a smooth function $u : M \to \mathbb{R}$ and construct the conformal metric $g^* = \mathrm{e}^{2u} g$. Then, given $E, F \in \mathcal{D}$, we have

$$(\mathcal{L}_K g^*)(E,F) = K(\mathrm{e}^{2u}) g(E,F) + \mathrm{e}^{2u}(\mathcal{L}_K g)(E,F) = 2\big(\rho_1 + K(u)\big) g^*(E,F).$$

A similar formula holds for $\mathcal{D}^\perp$. Therefore, K is also componentwise conformal when the metric $g^* = \mathrm{e}^{2u} g$ is considered on M. In addition, the associated symmetric tensor field α_K^* can be computed on $E \in \mathcal{D}$ and $F \in \mathcal{D}^\perp$ as follows,

$$\alpha_K^*(E,F) = (\mathcal{L}_K g^*)(E,F) = K(\mathrm{e}^{2u})g(E,F) + \mathrm{e}^{2u}(\mathcal{L}_K g)(E,F) = \mathrm{e}^{2u}\alpha_K(E,F),$$

that is to say, $\alpha_K^* = \mathrm{e}^{2u}\alpha_K$. In other words, the notion of componentwise conformal vector field on (M,g,P) is a conformal invariant.

A natural property to be required for a componentwise conformal vector field K is that all of its (local) flows commute with the almost product structure P.

Proposition 2.3. *Let K be a componentwise conformal vector field on (M,g,P). Then, the stages ψ_t of all (local) flows of K satisfy $(\psi_t)_* \circ P = P \circ (\psi_t)_*$ if, and only if,*

$$\mathcal{L}_K \widehat{P} = \lambda\, \widehat{P} + \mu\, g + \alpha_K(P,\ \ ). \tag{2.3}$$

Proof. If each ψ_t satisfies $(\psi_t)_* \circ P = P \circ (\psi_t)_*$, then it holds $(\mathcal{L}_K \widehat{P})(E,F) = (\mathcal{L}_K g)(P(E),F)$ for all $E,F \in \mathfrak{X}(M)$. Conversely, take $a,b \in T_pM$ and consider the real valued functions

$$f(t) = g((\psi_t)_*(a),(\psi_t)_*(b)), \quad \text{and} \quad h(t) = \widehat{P}((\psi_t)_*(a),(\psi_t)_*(b))$$

A standard argument from (2.2) and (2.3), respectively, shows that $f(t)$ and $h(t)$ have second derivatives

$$f''(t) = K_p(\lambda)g(a,b) + K_p(\mu)\widehat{P}(a,b), \text{ and } h''(t) = K_p(\lambda)\widehat{P}(a,b) + K_p(\mu)g(a,b).$$

Then,

$$f(t) = \frac{1}{2}\Big(K_p(\lambda)g(a,b) + K_p(\mu)\widehat{P}(a,b)\Big)\,t^2 + (\mathcal{L}_K g)(a,b)\,t + g(a,b).$$

$$h(t) = \frac{1}{2}\Big(K_p(\lambda)\widehat{P}(a,b) + K_p(\mu)g(a,b)\Big)\,t^2 + (\mathcal{L}_K \widehat{P})(a,b)\,t + \widehat{P}(a,b).$$

Therefore, $\widehat{P}((\psi_t)_*(a),(\psi_t)_*(b)) = g((\psi_t)_*(P(a)),(\psi_t)_*(b))$ for all $a,b \in T_pM$. □

Remark 2.6. For every Riemaniann almost product manifold (M,g,P), the tensor $\widehat{P}$ endows M with a semi-Riemannian metric. On the other hand, Definition 2.1 has an obvious extension to the semi-Riemannian case. A componentwise conformal vector field K on (M,g,P) satisfies (2.3), if and only if, K is also componentwise conformal for the semi-Riemannian metric $\widehat{P}$. On the other hand, it is a direct computation to check that the set of all componentwise conformal vector fields which satisfy (2.3) is a Lie algebra. This is not the situation for componentwise conformal vector fields in general (see Example 2.8.)

Remark 2.7. Observe that in our notion, no condition is imposed on $\mathcal{L}_K \widehat{P}$. A vector field K on a (semi)-Riemannian manifold (M,g) is said to be bi-conformal [6, Def. 3.1] when

$$\text{(a) } \mathcal{L}_K g = \lambda\, g + \mu\, \widehat{P} \quad \text{and} \quad \text{(b) } \mathcal{L}_K \widehat{P} = \lambda\, \widehat{P} + \mu\, g,$$

for $\lambda, \mu \in C^\infty(M)$. Thus, the notion of bi-conformal vector field is a very particular case of componentwise conformal vector field.

Example 2.8. Let $\mathbb{E}^2$ be the Euclidean plane with usual flat metric $g = dx^2 + dy^2$, with the almost product structure P given by $P(\partial_x) = \partial_x$ and $P(\partial_y) = -\partial_y$. Consider a vector field $K = a\,\partial_x + b\,\partial_y$, for some smooth functions a, b on $\mathbb{E}^2$. A direct computation shows

$$\mathcal{L}_K g = 2a_x\, dx \otimes dx + 2b_y\, dy \otimes dy + (a_y + b_x)\,(dx \otimes dy + dy \otimes dx).$$

In this case, any vector field K is componentwise conformal. Note that $\alpha_K = 0$ if and only if $a_y = -b_x$. By taking $K_1 = y\,\partial_x - x\,\partial_y$ and $K_2 = (x/3)\partial_x - (y/2)\partial_y$, we have that both K_1 and K_2 are componentwise conformal with $\alpha_{K_1} = \alpha_{K_2} = 0$. The symmetric tensor field corresponding to their Lie bracket satisfies $\alpha_{[K_1,K_2]} = (5/3)(dx \oplus dy + dy \oplus dx)$ and therefore, it never vanishes.

For the three dimensional Euclidean space $\mathbb{E}^3$ with its usual metric, consider the distribution $\mathcal{D} = \mathrm{Span}\{\partial_z\}$ with its corresponding tensor P. Next, let K be a vector field given by $K = a\,\partial_x + b\,\partial_y + c\,\partial_z$, for some smooth functions a, b, c in $\mathbb{E}^3$. We have that K is componentwise conformal if, and only if,

$$a_x = b_y \quad \text{and} \quad a_y = -b_x.$$

That is, for each $z \in \mathbb{R}$, the function $H^z(x+\mathbf{i}y) := a(x,y,z)+\mathbf{i}b(x,y,z)$ is holomorphic. Taking now $K_1 = (x - y + z)\partial_x + (x + y)\partial_y + x\,\partial_z$ and $K_2 = x\,\partial_z$, we have that K_1 and K_2 are componentwise conformal, but the Lie bracket $[K_1, K_2]$ is not (compare with [6, Prop. 5.2]).

Example 2.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let g be a left invariant Riemannian metric on G. Then, for every $E, F \in \mathfrak{g}$, the Levi-Civita connection ∇ of g satisfies [3, Prop. 3.18]

$$\nabla_E F = \frac{1}{2}\{[E,F] - (\mathrm{ad}_E)^*(F) - (\mathrm{ad}_F)^*(E)\},$$

where $(\mathrm{ad}_E)^*$ denotes the adjoint with respect to g of the linear map ad_E. Consider now an arbitrary element $K \in \mathfrak{g}$, then

$$\mathcal{L}_K g(E,F) = -\,g(\mathrm{ad}_K(E), F) - \,g(E, \mathrm{ad}_K(F)),$$

for $E, F \in \mathfrak{g}$. Assume $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^\perp$ where $\mathfrak{d}$ is any proper vector subspace of $\mathfrak{g}$ and consider the corresponding left invariant distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on G obtained from $\mathfrak{d}$ and $\mathfrak{d}^\perp$, respectively. Without loss of generality, we can consider $K \in \mathfrak{d}$. Therefore, whenever $\mathrm{ad}_K|_{\mathfrak{d}} = 0$ and $\mathrm{ad}_K|_{\mathfrak{d}^\perp} = c\,\mathrm{Id}$ with $c \neq 0$, the vector field K is a componentwise conformal vector field on (G, g, P), but not conformal, where P is the almost product structure corresponding to $\mathcal{D}$ and $\mathcal{D}^\perp$. For example, consider the subgroup G of the upper triangular matrices of the linear general group $Gl(n,\mathbb{R})$ given by $G = \{A \in Gl(n,\mathbb{R}) : a_{ij} = 0,\ \text{when } i > j\}$ with Lie algebra $\mathfrak{g} = \{E \in \mathfrak{gl}(n,\mathbb{R}) : a_{ij} = 0,\ i > j\}$. Define $\mathfrak{d} = \{E \in \mathfrak{g} : a_{1j} = 0,\ j \neq 1\}$ and $\mathfrak{f} = \{E \in \mathfrak{g} : a_{ij} = 0,\ \text{when } i > 1 \text{ and } a_{11} = 0\}$. That is,

$$\mathfrak{d} = \left\{ \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & \vdots & \ddots & a_{n-1\,n} \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \right\}, \quad \mathfrak{f} = \left\{ \begin{pmatrix} 0 & b_2 & \dots & b_n \\ 0 & 0 & \dots & 0 \\ 0 & \vdots & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\},$$

and take $K = (k_{ij}) \in \mathfrak{d}$ where $k_{11} = 1$ and $k_{ij} = 0$ otherwise. Let g be any left invariant Riemannian metric on G such that $\mathfrak{f} = \mathfrak{d}^{\perp}$. Therefore, $\mathrm{ad}_K|_{\mathfrak{d}} = 0$ and $\mathrm{ad}_K|_{\mathfrak{d}^{\perp}} = \mathrm{Id}$, which means that the left invariant vector field K is componentwise conformal on G.

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3 An integral formula

We will denote by ∇ the Levi-Civita connection of (M, g). Given a vector field $K \in \mathfrak{X}(M)$, we define the operator $L_K Y = -\nabla_Y K$, for any $Y \in \mathfrak{X}(M)$. With this notation, the Lie derivative takes the general form

$$(\mathcal{L}_K g)(E, F) \; = \; -g(L_K E, F) - g(E, L_K F),$$

for any $E, F \in \mathfrak{X}(M)$, $\mathrm{div}(K) = -\mathrm{Tr}L_K$ and the classical Bochner formula is written

$$K(\mathrm{Tr}L_K) = \mathrm{Ric}(K, K) - \mathrm{div}(\nabla_K K) + \mathrm{Tr}(L_K^2),$$

here Ric denotes the Ricci tensor of g. Making use of

$$\mathrm{div}(\mathrm{div}(K)K) = -K(\mathrm{Tr}L_K) + (\mathrm{Tr}L_K)^2,$$

when M is compact, we get,

$$\int_M \Big\{\mathrm{Ric}(K, K) + \mathrm{Tr}(L_K^2) - (\mathrm{Tr}L_K)^2\Big\}\mathrm{d}\mu_g = 0, \tag{3.1}$$

where $\mathrm{d}\mu_g$ denotes the canonical measure associated with g.

Theorem 3.1. *Let (M, g, P) be a compact Riemannian almost product manifold. Let $K \in \mathfrak{X}(M)$ be a componentwise conformal vector field. Then, we have*

$$\int_M \Big\{\mathrm{Ric}(K, K) + \frac{1}{2}\,\|\alpha_K\|^2 - \|\nabla K\|^2 + \Phi(\rho_1, \rho_2)\Big\}\mathrm{d}\mu_g = 0, \tag{3.2}$$

where α_K is the symmetric tensor field introduced in Lemma (2.1), *$\Phi(\rho_1, \rho_2) = n_1(2-n_1)\rho_1^2 + n_2(2-n_2)\rho_2^2 - 2n_1 n_2 \rho_1 \rho_2$, $n_1 = \mathrm{dim}\mathcal{D}$, $n_2 = \mathrm{dim}\mathcal{D}^{\perp}$ and ρ_1, ρ_2 are given in Definition* (2.1).

Proof. By assumption,

$$\begin{aligned} g(L_K E, F) + g(E, L_K F) = &- 2\rho_1 g(\pi(E), \pi(F)) - 2\rho_2 g(\pi^{\perp}(E), \pi^{\perp}(F)) \\ &- \alpha_K(E, F), \end{aligned}$$

for any $E, F \in \mathfrak{X}(M)$. Therefore,

$$L_K + L_K^t = -2\rho_1\pi - 2\rho_2\pi^{\perp} - \phi, \tag{3.3}$$

where L_K^t is the g-adjoint operator of L_K and ϕ is the g-self-adjoint operator defined by

$$\alpha_K(E,F) = g(\phi(E),F), \tag{3.4}$$

for any $E,F \in \mathfrak{X}(M)$. Directly from equation (3.3) we have

$$\mathrm{Tr}(L_K) = -\rho_1 n_1 - \rho_2 n_2. \tag{3.5}$$

Also from equation (3.3),

$$L_K^2 + (L_K^t)^2 + L_K L_K^t + L_K^t L_K = 4\rho_1^2\pi + 4\rho_1^2\pi^\perp + \phi^2 + 2\rho_1\phi\circ\pi + 2\rho_1\pi\circ\phi + 2\rho_2\,\phi\circ\pi^\perp + 2\rho_2\,\pi^\perp\circ\phi,$$

where we can take traces to get

$$2\mathrm{Tr}(L_K^2) + 2\|\nabla K\|^2 = 4\rho_1^2 n_1 + 4\rho_2^2 n_2 + \|\alpha_K\|^2, \tag{3.6}$$

because of $\mathrm{Tr}(\pi\circ\phi) = \mathrm{Tr}(\phi\circ\pi) = \mathrm{Tr}(\pi^\perp\circ\phi) = \mathrm{Tr}(\phi\circ\pi^\perp) = 0$. The proof concludes by inserting (3.5) and (3.6) in the general Bochner formula (3.1). □

Formula (3.2) can be seen as an extension to the one used by Yano [14] to analyse conformal vector fields on a compact Riemannian manifold under some curvature assumption [14, Th. 1]. In fact, the following consequence of previous theorem extends Yano's result.

Corollary 3.2. *Let (M,g,P) be an $(n \geq 3)$ compact Riemannian almost product manifold with nonpositive Ricci curvature. A componentwise conformal vector field K has vanishing covariant derivative whenever $\|\alpha_K\|^2 + 2\,\Phi(\rho_1,\rho_2) \leq 0$. Moreover, if the Ricci curvature is negative definite at some point, then K vanishes identically.*

As a consequence of Theorem (3.1), we reprove the following result in [12].

Corollary 3.3. *Let (M,g) be an $n(\geq 3)$-dimensional compact Riemannian manifold. If (M,g) admits an orthogonally conformal vector field Z, then*

$$\int_M \mathrm{Ric}(Z,Z)d\mu_g \geq 0. \tag{3.7}$$

The equality holds if, and only if, $\nabla_U Z = 0$ for any $U \perp Z$, and in such case, Z is orthogonally Killing (i.e., $\rho = 0$).

Proof. From Example 2.4, Z is componentwise conformal with $\rho_1 = 0$ and $\rho_2 = \rho$. Moreover, for every $E \in \mathfrak{X}(M)$, the operator ϕ given in (3.4) satisfies,

$$\phi(E) = g(E,Z)\nabla_Z Z + g(E,\nabla_Z Z)Z$$

and $\|\alpha_K\|^2 = \mathrm{Tr}(\phi^2) = 2\|\nabla_Z Z\|^2$. The integral formula (3.2) implies the announced inequality (3.7) and its equality condition. □

4 Applications to Riemannian submersions

Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion, and denote by $\mathbf{v}$ and $\mathbf{h}$ the (orthogonal) projections onto the vertical, $\mathcal{V}$, and horizontal, $\mathcal{H}$, distributions of p, respectively. Also, let A and T be the associated O'Neill tensors, [11]. In this section, we extensively make use of properties of tensors A and T (see for instance [1, Chap. 9] or [5, Chap. 1].) A direct computation gives,

Lemma 4.1. *Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion. Given $K \in \mathfrak{X}(B)$, let $\widehat{K} \in \mathfrak{X}(M)$ be its horizontal lift. Then it holds,*

$$(\mathcal{L}_{\widehat{K}}\widehat{g})(E,F) = (p^*\mathcal{L}_K g)(E,F) - 2\widehat{g}(T_{\mathbf{v}E}\mathbf{v}F, \widehat{K}) - \widehat{g}(A_E F + A_F E, \widehat{K}),$$

for any $E, F \in \mathfrak{X}(M)$.

Example 4.1. Now assume $p : (M, \widehat{g}) \to (B, g)$ is a Riemannian submersion with totally umbilical fibers. Consider the vertical distribution $\mathcal{D} = \mathcal{V}$ and its corresponding almost product structure P. In this case, the horizontal lift $\widehat{K} \in \mathfrak{X}(M)$ of a conformal vector field $K \in \mathfrak{X}(B)$ is componentwise conformal. Indeed, if $\mathcal{L}_K g = 2\rho\, g$ holds, then

$$(p^*\mathcal{L}_K g)(E,F) = 2\rho\, g(p_*(E), p_*(F)) = 2(\rho \circ p)\, \widehat{g}(\mathbf{h}E, \mathbf{h}F).$$

The tensor T evaluated on vertical vectors is just the second fundamental form II of the fibers, and therefore $\widehat{g}(T_{\mathbf{v}E}\mathbf{v}F, \widehat{K}) = \widehat{g}(\mathrm{II}(\mathbf{v}E, \mathbf{v}F), \widehat{K})$, for any $E, F \in \mathfrak{X}(M)$. If we assume the fibers are totally umbilical, their second fundamental forms are given by $\mathrm{II}(\mathbf{v}E, \mathbf{v}F) = \widehat{g}(\mathbf{v}E, \mathbf{v}F)\mathbf{H}$, where $\mathbf{H}$ is the mean curvature vector of the fibers. From Lemma 4.1, we have

$$(\mathcal{L}_{\widehat{K}}\widehat{g})(E,F) = 2\rho_1\, \widehat{g}(\mathbf{v}E, \mathbf{v}F) + 2(\rho \circ p)\widehat{g}(\mathbf{h}E, \mathbf{h}F) - \widehat{g}(A_E F + A_F E, \widehat{K}),$$

where, $\rho_1 = -\widehat{g}(\mathbf{H}, \widehat{K})$.

Example 4.2. Let $p : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ be the classical Hopf fibration. Take a nontrivial Killing vector field K on $\mathbb{S}^2$ and consider $\widehat{K}$ its horizontal lift, as in previous example. It is easy to see that $\rho_1 = \rho_2 = 0$ everywhere and $\alpha_{\widehat{K}} \neq 0$.

Next, in addition to the notations introduced in Example 4.1, we denote by $\widehat{\mathrm{Ric}}$, $\widehat{\nabla}$ and $\widehat{\nabla}^\perp$, respectively, the Ricci tensor of M, the Levi-Civita connection of M and the normal connection of the fibers.

Theorem 4.2. *Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion with M compact. Assume the fibers are totally umbilical and n_1-dimensional. Then, for every Killing vector field $K \in \mathfrak{X}(B)$,*

$$\begin{aligned} &\int_M \widehat{\mathrm{Ric}}(\widehat{K}, \widehat{K})\, \mathrm{d}\mu_{\widehat{g}} = \\ &\int_M \Big\{ \|\widehat{\nabla}^\perp \widehat{K}\|^2 + \|\nabla K\|^2 \circ p + n_1(n_1-1)\widehat{g}(\mathbf{H}, \widehat{K})^2 \Big\} \mathrm{d}\mu_{\widehat{g}}. \end{aligned} \tag{4.1}$$

Proof. From Example 4.1 we know that $\widehat{K}$ is componentwise conformal with $\rho_1 = -\widehat{g}(\mathbf{H}, \widehat{K})$, $\rho_2 = 0$ and the operator ϕ given in (3.4) satisfies

$$\phi E = A_{\mathbf{h}E}\widehat{K} + A_{\widehat{K}}\mathbf{v}E,$$

for every $E \in \mathfrak{X}(M)$. Let $\{U_1, \ldots, U_{n_1}, X_1, \ldots, X_{n_2}\}$ be a p-adapted local orthonormal frame. That is, the vector fields $U_i's$ span the vertical distribution $\mathcal{V}$, the $X_j's$ span the horizontal distribution $\mathcal{H}$ and are basic. Now, we compute the terms of integral formula (3.2), obtaining

$$\|\alpha_{\widehat{K}}\|^2 = \mathrm{Tr}(\phi^2) = 2\sum_{i=1}^{n_1} \|A_{\widehat{K}} U_i\|^2 = 2\sum_{i=1}^{n_1} \|\widehat{\nabla}^{\perp}_{U_i}\widehat{K}\|^2 = 2\|\widehat{\nabla}^{\perp}\widehat{K}\|^2.$$

On the other hand, since the fibers are totally umbilical, we get

$$\begin{aligned}\|\widehat{\nabla}\widehat{K}\|^2 &= \sum_{i=1}^{n_1} \|\widehat{\nabla}_{U_i}\widehat{K}\|^2 + \sum_{j=1}^{n_2} \|\widehat{\nabla}_{X_j}\widehat{K}\|^2 \\ &= n_1\widehat{g}(\mathbf{H}, \widehat{K})^2 + 2\|\widehat{\nabla}^{\perp}\widehat{K}\|^2 + \sum_{j=1}^{n_2} \|\mathbf{h}\widehat{\nabla}_{X_j}\widehat{K}\|^2 \\ &= n_1\widehat{g}(\mathbf{H}, \widehat{K})^2 + 2\|\widehat{\nabla}^{\perp}\widehat{K}\|^2 + \|\nabla K\|^2 \circ p.\end{aligned}$$

Therefore, Theorem (3.1) yields the announced integral formula (4.1). □

Corollary 4.3. *Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion with M compact. Assume the fibers are totally umbilical. Then, for every Killing vector field $K \in \mathfrak{X}(B)$, we have*

$$\int_M \widehat{\mathrm{Ric}}(\widehat{K}, \widehat{K})\, \mathrm{d}\mu_{\widehat{g}} \geq 0. \tag{4.2}$$

If $n_1 \geq 2$ (resp. $n_1 = 1$), the equality holds if, and only if, $\widehat{K}$ and K are parallel, (resp. $\widehat{\nabla}^{\perp}\widehat{K} = 0$ and K is parallel).

Remark 4.3. For every Killing vector field K on an arbitrary Riemannian manifold B, a well-known computation yields

$$\triangle \frac{1}{2}\|K\|^2 = \|\nabla K\|^2 - \mathrm{Ric}(K, K),$$

where $\triangle$ is the Laplacian of B. Therefore,

$$\int_B \mathrm{Ric}(K, K) \mathrm{d}\mu_g \geq 0, \tag{4.3}$$

and the equality holds if, and only if, K is parallel, [2]. Coming back to the previous situation $p : (M, \widehat{g}) \to (B, g)$, using [5, Chap. 1] and taking into account the umbilicity of the fibers, we have

$$\widehat{\mathrm{Ric}}(\widehat{K}, \widehat{K}) = \mathrm{Ric}(K, K) \circ p + n_1\widehat{g}(\widehat{\nabla}_{\widehat{K}}\mathbf{H}, \widehat{K}) - 2\|\widehat{\nabla}^{\perp}\widehat{K}\|^2 - n_1\widehat{g}(\mathbf{H}, \widehat{K})^2. \tag{4.4}$$

Therefore, the inequality in Corollary (4.3) cannot be deduced from the classical inequality (4.3).

Theorem 4.4. *Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion with totally umbilical fibers, where M is compact. Assume the Ricci tensor $\widehat{\mathrm{Ric}}$ of M is negative semidefinite on horizontal vectors. Then, every Killing vector field on B must be parallel.*

Remark 4.4. Recall now that the Lie algebra $\mathfrak{g}$ of the isometry group $\mathrm{Iso}(B)$ is naturally identified to the Lie algebra of the Killing vector fields on B, being B compact. Under the assumption of Theorem (4.4), the Lie algebra $\mathfrak{g}$ is abelian. Since B is compact, the Lie group $\mathrm{Iso}(B)$ is finite or its identity component is isomorphic to a k-dimensional torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Corollary 4.5. *Let $p : (M, \widehat{g}) \to (B, g)$ be a Riemannian submersion with totally umbilical fibers, where M is compact and B is homogeneous. Assume the Ricci tensor $\widehat{\mathrm{Ric}}$ of M is negative semidefinite on horizontal vectors. Then, the O'Neill tensor A vanishes (i.e., $\mathcal{H}$ is integrable). If moreover $n_1 \geq 2$ holds, then*

1. *The O'Neill tensor $T = 0$ (i.e., each fiber is totally geodesic),*
2. *B is isometric, up to a finite cover, to an n-dimensional flat torus.*

Proof. For every $q \in B$ and $v \in T_qB$, take a Killing vector field $K^v \in \mathfrak{X}(B)$ with $K^v_q = v$. The assumption on the Ricci tensor implies that equality holds in (4.2). Therefore $\widehat{\nabla}^{\perp}\widehat{K^v} = 0$ for all $q \in B$ and $v \in T_qB$. Now, it is not difficult to obtain that the O'Neill tensor A vanishes. If $n_1 \geq 2$, we get that $\widehat{K^v}$ is parallel and then $T = 0$. Hence, B must be Ricci flat from (4.4). Being B homogeneous, the result (2) follows from [8, Cor. 6.5.6]. □

Remark 4.5. Compare with [4, Prop. 3.1] where the author showed that a Riemannian submersion with totally geodesic fibers from a manifold M with nonpositive sectional curvature on Riemannian manifold B satisfies $A = 0$.

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