Criteria for Conformal Flatness of Finsler Spaces

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Abstract

The concept of conformal flatness of Finsler spaces is studied by several authors ([1], [2], [3], [5], [6], [7] and [8]). Especially, Kikuchi [6] obtained a very import ant theorem by using the function L^2C^2 under a certain condition. Hereafter, we shall call it the Kikuchi theorem.

The purpose of the present paper is to study some necessary and sufficient conditions for Finsler spaces to be conformally flat, which are similar to the Kikuchi theorem.

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1 Preliminaries

Let $F=(M,L)$ be an n-dimensional Finsler space endowed with fundamental function $L=L(x,y)$, where M is an n-dimensional differentiable manifold, $x=(x^i)$ is a point and $y=(y^i)$ is a supporting el ement of F, respectively. Then, the following notations are used

$$\begin{aligned}
&g_{ij}=(\partial L^2/\partial y^i\partial y^j)/2,\quad C_{ijk}=(\partial g_{ij}/\partial y^k)/2,\\
&C_i=g^{jk}C_{ijk},\quad C^2=g^{ij}C_iC_j,\quad g^{jk}=(g_{jk})^{-1},\\
&\gamma_j{}^i{}_k=g^{ir}(\partial g_{rk}/\partial x^j+\partial g_{rj}/\partial x^k-\partial g_{jk}/\partial x^r)/2,\\
&G^i=\gamma_j{}^i{}_k y^j y^k/2,\quad G^i{}_j=\partial G^i/\partial y^j,\\
&G^i{}_{jk}=\partial G^i{}_j/\partial y^k,\quad G^i{}_{jkl}=\partial G^i{}_{jk}/\partial y^l,\\
&F_j{}^i{}_k=\gamma_j{}^i{}_k-C_r{}^i{}_kG^r{}_j-C_r{}^i{}_jG^r{}_k+C_{rjk}G^r{}_tg^{ti},
\end{aligned}\tag{1.1}$$

where $C_r{}^i{}_k=C_{rtk}g^{ti}$ and C_{ijk} is called the $h(hv)$-torsion tensor. Hereafter, we adopt the Cartan connection $C\Gamma=(F_j{}^i{}_k,G^i{}_j,C_j{}^i{}_k)$ as a Finsler connection of F.

It is well-known that if all coefficients of $G^i{}_{jk}$ of an n-dimensional Finsler space F depend on position alone, then F is called a Berwald space, and the Berwald space is characterized by $C_{ijk|l}=0$, where the symbol $_|$ means the h-covariant derivative with

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respect to the Cartan connection $C\Gamma$. Moreover, if the fundamental function L of F depends on supporting element alone under a certain coordinate system (x^i), then F is called a locally Minkowski space.

Now we consider a change of fundamental function L to another fundamental function L^* on the same underlying manifold M, then we have two Finsler spaces $F = (M, L)$ and $F^* = (M, L^*)$. If there exists a scalar function $\sigma = \sigma(x)$ of position only such that $L^* = e^\sigma L$, then the change is called a *conformal change*. We introduce the following notations that will play important roles later

$$l^i = y^i/L, \quad B^{ij} = L^2(g^{ij} - 2l^i l^j)/2, \quad B^{ij}_k = \partial B^{ij}/\partial y^k,$$

$$B^{ij}_{km} = \partial B^{ij}_k/\partial y^m, \quad B^{ij}_{kml} = \partial B^{ij}_{km}/\partial y^l.$$

Hashiguchi [1] proved the following two propositions:
Proposition A. $B^{ij}, B^{ij}_k, B^{ij}_{km}$ *and* B^{ij}_{kml} *are invariant under any conformal change.*
Proposition B. *Under a conformal change* $L^* = e^\sigma L$*, the following relations are true*

$$g^*_{ij} = e^{2\sigma} g_{ij}, \quad g^{*ij} = e^{-2\sigma} g^{ij}, \tag{1}$$

$$G^{*i} = G^{*i} - B^{ir}\sigma_r, \quad G^{*i}{}_j = G^i{}_j - B^{ir}_j\sigma_r, \tag{2}$$

$$G^{*i}{}_{jk} = G^i{}_{jk} - B^{ir}_{jk}\sigma_r, \quad G^{*i}{}_{jkl} = G^i{}_{jkl} - B^{ir}_{jkl}\sigma_r, \tag{3}$$

$$C^*{}_j{}^i{}_k = C_j{}^i{}_k, \quad C^*{}_k = C_k, \tag{4}$$

where $\sigma_r = \partial\sigma/\partial x^r$.

2 The Kikuchi theorem and a similar theorem

In this section, we shall state conformally flat Finsler spaces which are introduced by the following definition:

Definition. An n-dimensional Finsler space $F = (M, L)$ with fundamental function L is called *conformally flat*, if for any point p of F, there exists a local coodinate neigborhood (U, x) containing p and a function $\sigma(x)$ on U, such that the Finsler space $F^* = (M, L^*)$ with fundamental function $L^* = e^\sigma L$ is a locally Minkowski space.

Kikuchi payed attension to the function L^2C^2 of a Finsler space F and introduced a tensor $W^i{}_j = (\partial L^2C^2/\partial y^r)B^{ri}_j$. Moreover, under the condition that the tensor $W^i{}_j$ is regular, he defined a vector B_j, a conformally invariant connection $M^i{}_{jk}$ and two conformally invariant curvature tensors $M^i{}_{jkl}$ and $H^i{}_{jkl}$ as follows

$$B_j = (L^2C^2)_{|k} W'^k{}_j, \text{ where } W'^k{}_j = (W^k_j)^{-1}, \tag{2.1}$$

(2.2) $$M^i{}_{jk} = G^i{}_{jk} - B^{ir}_{jk} B_r,$$

(2.3)
$$\begin{aligned} M^i{}_{jkl} &= \partial M^i{}_{jk}/\partial x^l - (\partial M^i{}_{jk}/\partial y^r) M^r{}_{lt} y^t - \partial M^i{}_{jl}/\partial x^k + (\partial M^i{}_{jl}/\partial y^r) M^r{}_{kt} y^t \\ &+ M^r{}_{jk} M^i{}_{rl} - M^r{}_{jl} M^i{}_{rk}, \end{aligned}$$

(2.4) $$H^i{}_{jkl} = G^i{}_{jkl} - B^{ir}_{jkl} B_r.$$

The Kikuchi Theorem [6]. *Let $F = (M, L)$ be an n-dimensional Finsler space for which $W^i{}_j$ is regular. The space F is conformally flat if and only if*

$$\partial B_i/\partial y^j = 0, \quad B_{i;j} - B_{j;i} = 0, \quad H^i{}_{jkl} = 0, \quad M^i{}_{jkl} = 0,$$

where the symbol ; means the h-covariant derivative with respect to the new Finsler connection $(M^i{}_{jk}, M^i{}_{jk} y^k)$.

We now consider two conditions that the function L^2C^2 in the Kikuchi theorem has to satisfy. The first condition is that L^2C^2 is conformally invariant and the second condition is that if F^* becomes a Berwald space under a conformal change $L^* = e^\sigma L$, then $L^{*2}C^{*2}$ is h-covariant constant with respect to the Cartan connection $C\Gamma^* = (F^*{}_j{}^i{}_k, G^{*i}{}_j, C^*{}_j{}^i{}_k)$ of F^*. So, if there exists a certain conformally invariant function A on an n-dimensional Finsler space F and satisfies the above second condition, then we get a theorem which is similar to the Kikuchi theorem exchanging the function L^2C^2 for this function A.

Since the function A is conformally invariant, that is $A = A^*$, from (1.1), it is derived that

(2.5) $$A^*{}_{|*k} = A_{|k} + A^r{}_k \sigma_r,$$

where $A^r{}_k = (\partial A/\partial y^i) B^{ir}_k$ and the symbol $_{|*}$ means h-covariant derivative with respect to the Cartan connection $C\Gamma^* = (F^*{}_j{}^i{}_k, G^{*i}{}_j, C^*{}_j{}^i{}_k)$ of F^*. Now, we assume that $A^r{}_k$ is regular tensor, then transvection of (2.5) by the reciprocal tensor $A'^r{}_j$ of $A^r{}_j$ yields

(2.6) $$\sigma_j = A_j - A^*{}_j,$$

where $A_j = -A_{|k} A'^k{}_j$, because $A'^r{}_j$ and $A^r{}_j$ are both conformally invariant. Put $A^i{}_{jk} = G^i{}_{jk} - B^{ir}_{jk} A_r$, then (3) of Proposition B and (2.6) show that $A^i{}_{jk}$ becomes a conformally invariant symmetric connection. Thus, we can get a conformally invariant curvature tensor

(2.7)
$$\begin{aligned} A^i{}_{jkl} &= \partial A^i{}_{jk}/\partial x^l - (\partial A^i{}_{jk}/\partial y^r) A^r{}_{lt} y^t - \partial A^i{}_{jl}/\partial x^k + \\ &+ (\partial A^i{}_{jl}/\partial y^r) A^r{}_{kt} y^t + A^r{}_{jk} A^i{}_{rl} - A^r{}_{jl} A^i{}_{rk}. \end{aligned}$$

Moreover, from (3) of Proposition B and (2.6), another conformally invariant curvature tensor $A'^i{}_{jkl}$ is definded by

$$A'^{i}{}_{jkl} = G^{i}{}_{jkl} - B^{ir}_{jkl} A_r. \tag{2.8}$$

Therefore, using a similar proof to those of Kikuchi [6], we have

Theorem 2.1. *Let $F = (M, L)$ be an n-dimensional Finsler space. If there exists a conformal invariant function A which satisfies the condition that $F^* = (M, L^*)$ is a Berwald space under a conformal change $L^* = e^{\sigma} L$, then A^* is h-covariant constant with respect to the Cartan connection $C\Gamma^*$ of F^* and the tensor $A^r{}_k = (\partial A/\partial y^i) B^{ir}_k$ is regular. Then F is conformally flat if and only if*

$$\partial A_i/\partial y^j = 0, \quad A_{i;j} - A_{j;i} = 0, \quad A^i{}_{jkl} = 0, \quad A'^{i}{}_{jkl} = 0,$$

where the symbol ; means the h-covariant derivative with respect to the new Finsler connection $(A^i{}_{jk}, A^i{}_{jk} y^k)$.

3 The conditions for a Finsler space to be conformally flat

In this section, we shall find some functions satisfying two conditions in Theorem 2.1. First, we state the $h(hv)$-torsion tensor C_{ijk}. From Proposition B, it is easily seen that $C^*{}_{ijk} = e^{2\sigma} C_{ijk}$ and $C^{*i} = e^{-2\sigma} C^i$, where $C^i = g^{ir} C_r$. Thus, a function $D = L^4 C_{ijk} C^i C^j C^k$ becomes conformally invariant. On the other hand, the h-covariant derivative of the function D^* vanishes, if the Finsler space F^* is a Berwald space. Therefore, we have

Theorem 3.1. *Let $F = (M, L)$ be an n-dimensional Finsler space with $det D^i{}_j \neq 0$. Then, a necessary and sufficient condition for F to be conformally flat is*

$$\partial D_i/\partial y^j = 0, \quad D_{i;j} - D_{j;i} = 0, \quad D^i{}_{jkl} = 0, \quad D'^{i}{}_{jkl} = 0,$$

where

$$\begin{aligned}
D &= L^4 C_{ijk} C^i C^j C^k, \quad D^r{}_k = (\partial D/\partial y^i) B^{ir}_k, \\
D_j &= -D_{|k} D'^{k}{}_j, \quad D'^{k}{}_j = (D^k{}_j)^{-1}, \quad D^i{}_{jk} = G^i{}_{jk} - B^{ir}_{jk} D_r, \\
D^i{}_{jkl} &= \partial D^i{}_{jk}/\partial x^l - (\partial D^i{}_{jk}/\partial y^r) D^r{}_{lt} y^t - \partial D^i{}_{jl}/\partial x^k + (\partial D^i{}_{jl}/\partial y^r) D^r{}_{kt} y^t + \\
&+ D^r{}_{jk} D^i{}_{rl} - D^r{}_{jl} D^i{}_{rk}, \\
D'^{i}{}_{jkl} &= G^i{}_{jkl} - B^{ir}_{jkl} D_r.
\end{aligned}$$

and the symbol ; means the h-covariant derivative with respect to the new Finsler connection $(D^i{}_{jk}, D^i{}_{jk} y^k)$.

Moreover, it is evident that a function $E = L^2 C_{ijk} C^{ijk}$ satisfies two conditions in Theorem 2.1. Therefore, we have

Theorem 3.2. *Let $F = (M, L)$ be an n-dimensional Finsler space with $det E^i{}_j \neq 0$. Then, a necessary and sufficient condition for F to be conformally flat is*

$$\partial E_i/\partial y^j = 0, \quad E_{i;j} - E_{j;i} = 0, \quad E^i{}_{jkl} = 0, \quad E'^{i}{}_{jkl} = 0,$$

where

$$\begin{aligned} E &= L^4 C_{ijk} C^{ijk},\ E^r{}_k = (\partial E/\partial y^i) B^{ir}_k, \\ E_j &= -E_{|k} D'^k{}_j,\ E'^k{}_j = (E^k{}_j)^{-1}, E^i{}_{jk} = G^i{}_{jk} - B^{ir}_{jk} E_r, \\ E^i{}_{jkl} &= \partial E^i{}_{jk}/\partial x^l - (\partial E^i{}_{jk}/\partial y^r) E^r{}_{lt} y^t - \partial E^i{}_{jl}/\partial x^k + (\partial E^i{}_{jl}/\partial y^r) E^r{}_{kt} y^t + \\ &+ E^r{}_{jk} E^i{}_{rl} - E^r{}_{jl} E^i{}_{rk}, \\ E'^i{}_{jkl} &= G^i{}_{jkl} - B^{ir}_{jkl} E_r. \end{aligned}$$

and the symbol ; means the h-covariant derivative with respect to the new Finsler connection $(E^i{}_{jk}, E^i{}_{jk} y^k)$.

Finally, we deal with the v-curvature tensor $S_{ijkl} (= C_{ilr} C_j{}^r{}_k - C_{ikr} C_j{}^r{}_l)$ of an n-dimensional Finsler space F. Iff $U = L^2 g^{il} g^{jk} S_{ijkl}$, then the function U satisfies two conditions of Theorem 2.1. So, we have

Theorem 3.3. *Let* $F = (M, L)$ *be an* n*-dimensional Finsler space with* $detU^i{}_j \neq 0$. *A necessary and sufficient condition for* F *to be conformally flat is that*

$$\partial U_i/\partial y^j = 0,\ \ U_{i;j} - U_{j;i} = 0,\ \ U^i{}_{jkl} = 0,\ \ U'^i{}_{jkl} = 0,$$

where

$$\begin{aligned} U &= L^2 g^{il} g^{jk} S_{ijkl},\ U^r{}_k = (\partial U/\partial y^i) B^{ir}_k, \\ U_j &= -E_{|k} D'^k{}_j,\ \ U'^k{}_j = (E^k{}_j)^{-1},\ \ U^i{}_{jk} = G^i{}_{jk} - B^{ir}_{jk} U_r, \\ U^i{}_{jkl} &= \partial U^i{}_{jk}/\partial x^l - (\partial U^i{}_{jk}/\partial y^r) U^r{}_{lt} y^t - \partial U^i{}_{jl}/\partial x^k + (\partial U^i{}_{jl}/\partial y^r) U^r{}_{kt} y^t + \\ &+ U^r{}_{jk} U^i{}_{rl} - U^r{}_{jl} U^i{}_{rk}, \\ U'^i{}_{jkl} &= G^i{}_{jkl} - B^{ir}_{jkl} U_r. \end{aligned}$$

and the symbol ; means the h-covariant derivative with respect to the new Finsler connection $(E^i{}_{jk}, E^i{}_{jk} y^k)$.

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