Spherically Symmetric Generalized Lagrange Metrics

Valentin Gîrţu and Monica Gîrţu

Abstract

Spherically symmetric metrics in the Finslerian setting are studied. Taking a different aproach from that of G.S. Asanov [3], the main geometrical objects associated to such a metric are found. The Einstein equations in vacuum are written.

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Introduction

In the last decade a revision and a generalization of the methods of general relativities were done by replacing the Riemannian metrics with more general ones.

As a first step a Finslerian theory of relativity was proposed and accordingly the Finslerian-Einstein equations were derived. Such equations were proposed on three quite distinct ways. The first one is a subtle combination between physical intuition and formal aspects and has its roots in the Yukawa bilocal theory, see [9],[6],[5]. The second one is provided by variational principles and extends the Palatini method, see [2],[3]. The third, of geometrical nature, was suggested by the involvement of tangent bundle in the Finsler geometry (see [8] and references therein).

These ways have advantages and disadvantages but only if they are put together a complete picture of the whole theory is obtained.

Our contribution in this paper is on the third line mentioned in the above. First, we have to stress that though this line of development is more geometric, there exist many physical arguments which support it, see [5]. Besides, its generality produces a richness of physical interpretations and such as its usefulness increases.

In a synthetic and rather vague way it can be described as follows. One considers a smooth manifold M of finite dimension and the tangent manifold TM fibered over M by the usual projection $\tau : TM \to M$. A symmetric and nondegenerate tensor field on TM, whose local components $(g_{ij}(x,y))$ behave like the components of a similar tensor field on M is called a *generalized Lagrange metric*, briefly a GL-metric. Assuming that the vertical distribution on TM, i.e. the kernel of the tangent map τ^T, has a supplementary distribution (horizontal), usually called a *nonlinear connection*,

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a pseudo–Riemannian metric G on TM is derived from $g_{ij}(x,y)$. Then it is shown that a linear connection D which is metrical with respect to G and preserve both distributions exists. The generalized Einstein equations are just the usual Einstein equations written for the pair (G, D) and an arbitrary energy–momentum tensor field. A similar construction can be done for a vector bundle. For the details, we refer to [8].

As it is well–known, in the general theory of relativity and in cosmologies several categories of pseudo–Riemannian metrics are used. Among them of great interest are the so–called (static) spherically symmetric metrics. A generalization of these metrics to the Finslerian setting was proposed and studied by G.S. Asanov [2],[3]. In spite of the various assumptions on homogeneity he made, the metric he proposed is not Finslerian but a GL-metric. This fact, recognized by Asanov himself, produces a difficulty since, in general, it is not possible to determine a nonlinear connection from a GL-metric as it happens for Finslerian metrics. G.S. Asanov overcomes this difficulty by assuming the vanishing of the h-deflection tensor of D. Thus he succeded to determine the local coefficients of D ingeniously solving a complicated equation involving these coefficients.

In the following we shall adopt a different point of view in order to develop the geometry of a GL-metric which is a slight generalization of Asanov's metric. As this GL-metric is constructed using a pseudo–Riemannian metric $(r_{ij}(x))$ on M, we use $(r_{ij}(x))$ for determining a nonlinear connection and so the finding of the local coefficients of D is merely a problem of calculation.

The paper is organized as follows. In §1 we state the GL-metric to be studied and we find the nonlinear connection to be used later. In §1 the linear connection D is determined by its local coefficients. Its torsions and deflection tensors are computed. In §3 the curvatures of D are computed and the generalized Einstein equations in vacuum are written. Some concluding remarks end the paper.

1 A static spherically symmetric GL-metric

Let M be a smooth n-dimensional manifold and (x^i), $i, j, ... = 1, ..., n$ its local coordinates. We shall denote by (x^i, y^i) the local coordinates on the tangent manifold TM such that $\tau : (x^i, y^i) \to x^i$.

A change of coordinates $(x^i, y^i) \to (\tilde{x}^i, \tilde{y}^i)$ on TM has the form

$$\begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n), \text{ rank } \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n, \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^k}(x) y^k. \end{aligned} \tag{1.1}$$

The local components $(g_{ij}(x,y))$ of a GL-metric satisfy

$$\begin{aligned} &g_{ij}(x,y) = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^h}{\partial x^j}\tilde{g}_{kh}(\tilde{x}(x), \tilde{y}(x,y)), \\ &g_{ij}(x,y) = g_{ji}(x,y), \quad \det(g_{ij}(x,y)) \neq 0. \end{aligned} \tag{1.2}$$

The smoothness of class C^4 will be assumed.

Assume that a symmetric nondegenerate tensor $r_{ij}(x)$ is given on M. It is static spherically symmetric (see (13.22) in [7]) if the coordinates (x^0, x^α), $\alpha, \beta, \gamma = 1, ..., n-1$ can be introduced on M, such that

$$r_{00} = r_{00}(r),\ r_{\alpha\beta} = -W(r)\delta_{\alpha\beta},\ r_{\alpha 0} = 0, \tag{1.3}$$

where $\delta_{\alpha\beta}$ stand for the Kronecker symbols, $r_{00} > 0, W \neq 0, r = (\overset{\Sigma}{\alpha}(x^\alpha)^2)^{\frac{1}{2}}$. These coordinates are called *isotropic*. The tensor $r_{ij}(x)$ may be thought of as a particular GL-metric. From now on we take $n = 4$.

One considers four positive scalars A_i, $i = 0, 1, 2, 3$ on TM and one constructs a GL-metric $(g_{ij}(x, y))$ putting

$$g_{ii}(x,y) = \rho_i(x,y), \qquad g_{ij} = 0 \text{ for } i \neq j, \text{ with} \tag{1.4}$$

$$\rho_0(x,y) = A_1(x,y) r_{00}(r), \quad \rho_\alpha(x,y) = -A_\alpha(x,y) W(r). \tag{1.5}$$

Later we shall assume that the functions A_i depend only on r and q where

$$q = -\sum_{\alpha,\beta} r_{\alpha\beta} y^\alpha y^\beta / [r_{00}(y^0)^2]^{-1}. \tag{1.6}$$

The GL-metric obtained in such a way will be called static spherically symmetric. The metric considered by G.S. Asanov in [3] is obtained from (1.4) with $A_0 \neq A_1 = A_2 = A_3$ depending on r and q only.

We notice that $\varepsilon = \sum_{i,j} r_{ij} y^i y^j$ is the so–called *absolute energy* of the GL–metric $(r_{ij}(x))$. It is clear that the function q was choosen such that to be homogeneous of degree zero. As in our considerations the homogeneity has no any role, it is more reasonable to assume that A_i depend only on r and ε.

It is obvious that $(g_{ij}(x, y))$ given by (1.4) and (1.5) is a GL-metric. In general, a GL-metric (g_{ij}) is said to be reducible to a Lagrange metric, shortly an L-metric if there exists a function $L : TM \to R$ such that $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 L(x,y)}{\partial y^i \partial y^j}\cdot$ If L is homogeneous of degree 2 with respect to y, $(g_{ij}(x, y))$ becomes a Finslerian metric. It is our GL-metric (1.4)-(1.5) reducible to an L-metric or to a Finslerian one? For answering this question one associates to $(g_{ij}(x, y))$ a d-tensor field of components $C_{ijk}(x, y) = = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^k}$ and by the Proposition 1.1, Ch.X in [8], the GL-metric $g_{ij}(x, y)$ is reducible to an L-metric if $C_{ijk}(x, y) \neq 0$ is totally symmetric. We have

$$C_{ijk}(x,y) = 0 \text{ for } i \neq j, \qquad C_{ijk}(x,y) = \frac{1}{2}\frac{\partial \rho_i}{\partial y^k}\cdot \tag{1.7}$$

It results that $C_{ijk} \neq 0$ when ρ_i depend on y. If this tensor field would be totally symmetric, then $\dfrac{\partial \rho_i}{\partial y^k} = \dfrac{\partial \rho_k}{\partial y^i}$ a equality which is not true for arbitrary A_i. Thus, the GL-metric (g_{ij}) given by (1.4)-(1.5) is not a Lagrange metric nor a Finslerian one.

Let $V_u TM = \ker \tau_u^T$, $u \in TM$, be the vertical subspace of $T_u TM$. The vertical distribution $u \to V_u TM$, $u \in TM$ is integrable and is locally spanned by $\left(\dfrac{\partial}{y^i}\right)\cdot$ A nonlinear connection N is a distribution $u \to H_u TM$, called horizontal distribution, which is supplementary to the vertical distribution, i.e.,

$$H_uTM \oplus V_uTM = T_uTN \quad \text{holds.} \tag{1.8}$$

The horizontal distribution is spanned by n local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k(x,y)\frac{\partial}{y^k},$$

where the functions $(N_i^k(x,y))$ have to satisfy

$$\tilde{N}_j^h \frac{\partial \tilde{x}^j}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^k} N_i^k - \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k} y^k, \tag{1.9}$$

when a change of coordinates (1.1) on TM is performed. These functions are called the local coefficients of the nonlinear connection N.

Coming back to $(r_{ij}(x))$, let $\rho^i_{jk}(x)$ be the Christoffel symbols

$$\rho^i_{jk} = \frac{1}{2} r^{ih}\left(\frac{\partial r_{jk}}{\partial x^k} + \frac{\partial r_{kh}}{\partial x^j} - \frac{\partial r_{jk}}{\partial x^h}\right). \tag{1.10}$$

Proposition 1.1. *The functions*

$$N^i_j(x,y) = \sum_k \rho^i_{jk}(x) y^k \tag{1.11}$$

are the local coefficients of a nonlinear connection N.

Proof. By using the usual law of transformation for $\rho^i_{jk}(x)$ one easily checks (1.9) for the given functions.

Now we shall seek for an explicit form of $(N^i_j(x,y))$. The tensor field $(r_{ij}(x))$ is of the form

$$r_{ii} = r_i, \qquad r_{ij} = 0 \qquad \text{for } i \neq j, \tag{1.12}$$

$$r_0 = r_{00}(r), \quad r_1 = r_2 = r_3 = -W(r). \tag{1.12$'$}$$

Its reciprocal (r^{ij}) has the form

$$r^{ii} = \frac{1}{r_i},\ r^{ij} = 0 \text{ for } i \neq j. \tag{1.13}$$

Inserting (1.12)–(1.13) in (1.10), one gets

$$\begin{aligned}
&\rho^i_{jk}(x) = 0 \text{ for } i \neq j \neq k \neq i,\\
&\rho^i_{jk}(x) = \frac{1}{2r_i}\frac{\partial r_i}{\partial x^k} \text{ for } k \neq i,\\
&\rho^i_{jj}(x) = -\frac{1}{2r_i}\frac{\partial r_j}{\partial x^i} \text{ for } i \neq j,\\
&\rho^i_{ii}(x) = \frac{1}{2r_i}\frac{\partial r_i}{\partial x^i}.
\end{aligned} \tag{1.14}$$

Then (1.11) leads to the following

$$N_j^i(x,y) = \frac{1}{2r_i}\left(\frac{\partial r_i}{\partial x^j}y^i - \frac{\partial r_j}{\partial x^i}y^j\right) \text{ for } i \neq j,$$
$$N_i^i(x,y) = \frac{1}{2r_i}\sum_k \frac{\partial r_i}{\partial x^k}y^k. \tag{1.15}$$

The d-tensor field $\tau_{jk}^i = \dfrac{\partial N_k^i}{\partial y^j} - \dfrac{\partial N_j^i}{\partial y^k}$ is called the torsion of the nonlinear connection $N(N_j^i)$. In our case, $\dfrac{\partial N_j^i}{\partial y^k} = \rho_{jk}^i x$ and so we have

Proposition 1.2. *The nonlinear connection of local coefficients* (1.11) *has no torsion. The curvature of the nonlinear connection* $N(N_j^i)$ *is given by the d tensor field of components*

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}. \tag{1.16}$$

Owing to (1.11), it follows directly that

$$R_{jk}^i(x,y) = \sum_k \rho_h{}^i{}_{jk}(x) y^h, \tag{1.17}$$

where $\rho_h{}^i{}_{jk}$ is the curvature of the metric $(r_{ij}(x))$. Thus, in order to compute $R_{jk}^i(x,y)$ we need to compute the curvature $\rho_h{}^i{}_{jk}(x)$. We notice that $R_{jk}^i = 0$ if and only if the metric $(r_{ij}(x))$ has no curvature.

As it is well-known, the curvature tensor of $(r_{ij}(x))$ is

$$\rho_j{}^i{}_{kh}(x) = \frac{\partial \rho_{jk}^i}{\partial x^h} - \frac{\partial \rho_{jh}^i}{\partial x^k} + \sum_s \rho_{jk}^s \rho_{sh}^i - \sum_s \rho_{jh}^i \rho_{sk}^i. \tag{1.18}$$

Inserting $\rho_{jk}^i(x)$ from (1.14) in (1.18), after a long calculation, one gets

$$\begin{aligned}
&\rho_j{}^i{}_{kh} = 0 \text{ for } i \neq j \neq k \neq h \neq i;\ \rho_i{}^i{}_{hk} = 0 \text{ for } k \neq i, h \neq i;\ \rho_i{}^i{}_{ih} = 0 \text{ for } i \neq h;\\
&\rho_j{}^i{}_{ih} = \frac{1}{4r_i}\left[2\frac{\partial^2 r_i}{\partial x^j \partial x^h} - \frac{1}{r_i}\frac{\partial r_i}{\partial x^j}\frac{\partial r_i}{\partial x^h} - \frac{1}{r_j}\frac{\partial r_j}{\partial x^h}\frac{\partial r_i}{\partial x^j} - \frac{1}{r_h}\frac{\partial r_h}{\partial x^j}\frac{\partial r_i}{\partial x^h}\right] \\
&\hspace{20em} \text{for } i \neq h \neq i,\\
&\rho_j{}^i{}_{kj} = \frac{1}{4r_i}\left[2\frac{\partial^2 r_j}{\partial x^i \partial x^k} - \frac{2}{r_i}\frac{\partial r_i}{\partial x^k}\frac{\partial r_j}{\partial x^i} - \frac{1}{r_j}\frac{\partial r_j}{\partial x^h}\frac{\partial r_j}{\partial x^i}\right] \quad \text{for } i \neq j \neq k \neq i,\\
&\rho_j{}^i{}_{ij} = \frac{1}{2r_i}\frac{\partial^2 r_i}{(\partial x^j)^2} - \frac{1}{2r_i^2}\left(\frac{\partial r_i}{\partial x^j}\right)^2 - \frac{1}{2r_i^2}\frac{\partial r_i}{\partial x^i}\frac{\partial r_j}{\partial x^i} + \frac{1}{2r_i}\frac{\partial^2 r_j}{(\partial x^i)^2} -\\
&\quad - \frac{1}{4r_i r_j}\left(\frac{\partial r_j}{\partial x^i}\right)^2 + \frac{1}{4r_i^2}\left(\frac{\partial r_i}{\partial x^j}\right)^2 - \frac{1}{4r_i r_j}\frac{\partial r_i}{\partial x^j}\frac{\partial r_j}{\partial x^j} +\\
&\quad + \frac{1}{4r_i^2}\frac{\partial r_j}{\partial x^i}\frac{\partial r_i}{\partial x^i} + \sum_{s \neq j}\frac{1}{r_i r_s}\frac{\partial r_j}{\partial x^s}\frac{\partial r_i}{\partial x^s} \hspace{8em} \text{for } i \neq j.
\end{aligned} \tag{1.19}$$

Now, owing to (1.17), from (1.19), it results

$$(1.20)\qquad \begin{aligned} R^i_{kh} &= \rho_k{}^i_{kh} y^k + \rho_h{}^i{}_{kh} y^h = \frac{y^h}{4r_i}\left(2\frac{\partial^2 r_h}{\partial x^i \partial x^k} - \frac{2}{r_i}\frac{\partial r_i}{\partial x^k}\frac{\partial r_h}{\partial x^i} - \frac{1}{r_h}\frac{\partial r_h}{\partial x^k}\frac{\partial r_h}{\partial x^i}\right) - \\ &\quad - \frac{y^k}{4r_i}\left(2\frac{\partial^2 r_k}{\partial x^i \partial x^h} - \frac{2}{r_i}\frac{\partial r_i}{\partial x^h}\frac{\partial r_h}{\partial x^i} - \frac{1}{r_k}\frac{\partial r_k}{\partial x^h}\frac{\partial r_k}{\partial x^i}\right) \text{ for } i\neq k\neq h\neq i \\ R^i_{ih} &= \sum_{j\neq i} \frac{1}{4r_i}\left(2\frac{\partial^2 r_i}{\partial x^j \partial x^h} - \frac{1}{r_i}\frac{\partial r_i}{\partial x^j}\frac{\partial r_i}{\partial x^h} - \frac{1}{r_j}\frac{\partial r_j}{\partial x^h}\frac{\partial r_i}{\partial x^j} - \frac{1}{r_h}\frac{\partial r_h}{\partial x^j}\frac{\partial r_i}{\partial x^h}\right) y^j. \end{aligned}$$

Concluding, we have found a natural nonlinear connection (1.11) without torsion and with the curvature (1.20).

2 Metrical linear connection of a static spherically symmetric GL-metric

In [8, ch.X], one associates to a GL-metric (g_{ij}) a metrical linear connection $C\Gamma(N)$ of local coefficients $(N^i_j(x,y), L^i_{jk}(x,y), C^i_{jk}(x,y))$, where

$$(2.1)\qquad L^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{jh}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h}\right),\ C^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h}\right)\cdot$$

The connection $C\Gamma(N)$ preserves the decomposition (1.8) and is metrical, i.e.

$$(2.2)\qquad g_{ij|k} = 0,\ g_{ij}\Big|_k = 0,$$

where short (resp. long) vertical bar stands for covariant derivative with respect to L^i_{jk} (resp. C^i_{jk}). Two torsions of $C\Gamma(N)$ vanish, and this fact corresponds to the symmetry of L^i_{jk} and C^i_{jk} in j and k.

We seek for an explicit form of $C\Gamma(N)$ for our GL-metric (1.4)-(1.5). First, we notice that L^i_{jk} and C^i_{jk} have the same form as ρ^i_{jk} but only the operators $\dfrac{\partial}{x^k}$ are replaced by $\dfrac{\delta}{x^k}$ and $\dfrac{\partial}{y^k}$, respectively. Accordingly, L^i_{jk} and C^i_{jk} will be obtained from (1.14) by replacing r_i by ρ_i and the operators $\dfrac{\partial}{x^k}$ by $\dfrac{\delta}{x^k}$ and $\dfrac{\partial}{y^k}$, respectively. Thus, we get

$$(2.3)\qquad \begin{aligned} &L^i_{jk} = 0 \text{ for } i\neq j\neq k\neq i, \quad L^i_{ik} = \frac{1}{2\rho_i}\frac{\delta\rho_i}{\delta x^k} \text{ for } i\neq k, \\ &L^i_{jj} = -\frac{1}{2\rho_i}\frac{\delta\rho_j}{\delta x^i} \text{ for } i\neq j, \quad L^i_{ii} = \frac{1}{2\rho_i}\frac{\delta\rho_i}{\delta x^i}\cdot \end{aligned}$$

Similarly, one obtains:

$$(2.4)\qquad \begin{aligned} &C^i_{jk} = 0 \text{ for } i\neq j\neq k\neq i, \quad C^i_{ik} = \frac{1}{2\rho_i}\frac{\partial\rho_i}{\partial y^k} \text{ for } i\neq k \\ &C^i_{jj} = -\frac{1}{2\rho_i}\frac{\partial\rho_j}{\partial y^i} \text{ for } i\neq j, \quad C^i_{ii} = \frac{1}{2\rho_i}\frac{\partial\rho_i}{\partial y^i}\cdot \end{aligned}$$

The other torsions of $C\Gamma(N)$ are $C^i_{jk}(x,y)$, R^i_{jk} and P^i_{jk}, where

$$P^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i_{jk}. \tag{2.5}$$

In our case, $P^i_{jk} = \rho^i_{jk} - L^i_{jk}$, hence it is symmetric in j and k. An explicit form of it can be infered from (2.3) and (1.14). From such a form one easily concludes that (P^i_{jk}) vanishes if ρ_i do not depend on y.

The h-deflection tensor of $C\Gamma(N)$ is the d-tensor field

$$D^i_j = y^i{}_{|j} = L^i_{kj}(x,y)y^k - N^i_j(x,y) \tag{2.6}$$

and the v-deflection tensor is

$$d^i_j = y^i\big|_j = \delta^i_j + C^i_{kj}(x,y)y^k. \tag{2.7}$$

Remark 2.1. For the GL-metric (1.4)-(1.5) we have $D^i_j = -y^k P^i_{kj}$.

The deflection tensor field are involved in the theory of electromagnetism proposed in [8, ch.X].

A first form of these tensor fields is as follows

$$\begin{aligned} D^i_j &= \frac{y^j}{2\rho_i}\sum_s N^s_i \frac{\partial \rho_j}{\partial y^s} - \frac{y^i}{2\rho_i}\sum_s N^s_j \frac{\partial \rho_i}{\partial y^s} \quad \text{for } i \neq j, \\ D^i_i &= \frac{y^i}{2\rho_i}\sum_s N^s_i \frac{\partial \rho_i}{\partial y^s} - \frac{1}{2\rho_i}\sum_{k\neq i}\left(\sum_s N^s_k \frac{\partial \rho_i}{\partial y^s}\right) y^k. \end{aligned} \tag{2.8}$$

$$\begin{aligned} d^i_j &= \frac{1}{2\rho_i}\left(y^i \frac{\partial \rho_i}{\partial y^j} - y^j \frac{\partial \rho_i}{\partial y^i}\right) \quad \text{for } i \neq j, \\ d^i_i &= 1 + \frac{1}{2\rho_i}\sum_s y^s \frac{\partial \rho_i}{\partial y^s}, \end{aligned} \tag{2.9}$$

where (N^i_j) is given by (1.15).

In the theory of electromagnetism mentioned in the above, the h-electromagnetic field is defined as the skewsymmetric part of $D_{ij} = \sum_k g_{ik} D^k_j$ while the v-electromagnetic field is defined as the skewsymmetric part of $d_{ij} = \sum_k g_{ik} d^k_j$.

In our case, (2.8) leads to

$$F_{ij} = \frac{1}{4}\left[y^j\left(\sum_s N^s_i \frac{\partial \rho_i}{\partial y^s}\right) - y^i\left(\sum_s N^s_j \frac{\partial \rho_i}{\partial y^s}\right)\right], \tag{2.10}$$

$$f_{ij} = \frac{1}{4}\left(y^i \frac{\partial \rho_i}{\partial y^s} - y^j \frac{\partial \rho_j}{\partial y^i}\right), \tag{2.11}$$

where again the functions N^i_j are given by (1.15).

3 Curvatures of $C\Gamma(N)$. The Einstein equations in vacuum

The connection $C\Gamma(N)$ has three curvatures

$$
\begin{aligned}
R_j{}^i{}_{kh}(x,y) &= \frac{\delta L^i_{jk}}{\delta x^h} - \frac{\delta L^i_{jh}}{\delta x^k} + \sum_s L^s_{jk}L^i_{sh} - \sum_s L^s_{jh}L^i_{sk} + \sum_s C^i_{js}R^s_{kh},\\
P_j{}^i{}_{kh}(x,y) &= \frac{\partial L^i_{jk}}{\partial y^h} - C^i_{jh|k} + \sum_s C^i_{js}P^s_{kh},\\
S_j{}^i{}_{kh}(x,y) &= \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + \sum_s C^s_{jk}C^i_{sh} - \sum_s C^s_{jh}C^i_{sk}.
\end{aligned}
$$

The following notation will be used

$$
\begin{aligned}
&R_{jk} = \sum_i R_j{}^i{}_{ki},\ \overset{1}{P}_{jk} = \sum_i P_j{}^i{}_{ki},\ \overset{2}{P}_{jk} = \sum_i P_j{}^i{}_{ik},\\
&S_{jk} = \sum_i S_j{}^i{}_{ki},\ R = g^{jk}R_{jk},\ S = g^{jk}S_{jk}.
\end{aligned}
\tag{3.1}
$$

The Einstein equations associated to $C\Gamma(N)$ are as follows (see §3,[8, ch.V])

$$
\begin{aligned}
&R_{ij} - \frac{1}{2}Rg_{ij} = \kappa \overset{H}{\mathcal{T}}_{ij}, \quad \overset{1}{P}_{jk} = \kappa \overset{1}{\mathcal{T}}_{jk}\\
&S_{ij} - \frac{1}{2}Sg_{ij} = \kappa \overset{V}{\mathcal{T}}_{ij}, \quad \overset{2}{P}_{jk} = \kappa \overset{2}{\mathcal{T}}_{jk}\ .
\end{aligned}
\tag{3.2}
$$

In the right hand of (3.2) the components of the energy–momentum tensor field appear.

In vacuum ($k = 0$) the equations (3.2) reduce to

$$
R_{ij} = 0,\ S_{ij} = 0,\ \overset{1}{P}_{ij} = 0,\ \overset{2}{P}_{ij} = 0. \tag{3.3}
$$

Now we shall seek for the explicit forms of the equations (3.3). We deal with the first two only as most important. The second is easier. We shall begin with it. Noticing that $S_j{}^i{}_{kh}$ has the same form as $\rho_j{}^i{}_{kh}$ but $\frac{\partial}{x^k}$ is replaced with $\frac{\partial}{y^k}$, the following form of it is directly infered from (1.19):

(3.4)
$$S_j{}^i{}_{kh}=0 \text{ for } i\neq j\neq k\neq h\neq i,\ S_i{}^i{}_{kh}=0 \text{ for } k\neq i, h\neq i,\ S_i{}^i{}_{ih}=0 \text{ for } i\neq h,$$
$$S_j{}^i{}_{ih}=\frac{1}{4\rho_i}\left[2\frac{\partial^2\rho_i}{\partial y^j\partial y^h}-\frac{1}{\rho_i}\frac{\partial\rho_i}{\partial y^j}\frac{\partial\rho_i}{\partial y^h}-\frac{1}{\rho_j}\frac{\partial\rho_j}{\partial y^h}\frac{\partial\rho_i}{\partial y^j}-\frac{1}{\rho_h}\frac{\partial\rho_h}{\partial y^j}\frac{\partial\rho_i}{\partial y^h}\right]\quad \text{for } i\neq j\neq h\neq i,$$
$$S_j{}^i{}_{kj}=\frac{1}{4\rho_i}\left[2\frac{\partial^2\rho_j}{\partial y^i\partial y^k}-\frac{2}{\rho_i}\frac{\partial\rho_i}{\partial y^k}\frac{\partial\rho_j}{\partial y^i}-\frac{1}{\rho_j}\frac{\partial\rho_j}{\partial y^k}\frac{\partial\rho_j}{\partial y^i}-\right]\quad \text{for } i\neq j\neq k\neq i,$$
$$S_j{}^i{}_{ij}=\frac{1}{2\rho_i}\frac{\partial^2\rho_i}{(\partial y^j)^2}-\frac{1}{2\rho_i^2}\left(\frac{\partial\rho_i}{\partial y^j}\right)^2-\frac{1}{2\rho_i^2}\frac{\partial\rho_i}{\partial y^i}\frac{\partial\rho_j}{\partial y^i}+\frac{1}{2\rho_i}\frac{\partial^2\rho_j}{(\partial y^i)^2}-$$
$$-\frac{1}{4\rho_i\rho_j}\left(\frac{\partial\rho_j}{\partial y^i}\right)^2+\frac{1}{4\rho_i^2}\left(\frac{\partial\rho_i}{\partial y^j}\right)^2-\frac{1}{4\rho_i\rho_j}\frac{\partial\rho_i}{\partial y^j}\frac{\partial\rho_j}{\partial y^j}+$$
$$+\frac{1}{4\rho_i^2}\frac{\partial\rho_j}{\partial y^i}\frac{\partial\rho_i}{\partial y^i}+\sum_{s\neq j}\frac{1}{4\rho_i\rho_s}\frac{\partial\rho_j}{\partial y^s}\frac{\partial\rho_i}{\partial y^s}\quad \text{for } i\neq j.$$

Then we have

(3.5)
$$S_{jk}=-\sum_{i\neq j,\, i\neq k}S_j{}^i{}_{ik}\ \text{ for } j\neq k,\quad S_{jj}=-\sum_{i\neq j}S_j{}^i{}_{ij}.$$

(3.6)
$$S=-\sum_j\sum_{i\neq j}\rho^j S_j{}^i{}_{ij}.$$

Now, if we set

(3.7)
$$\tilde{R}_j{}^i{}_{kh}=R_j{}^i{}_{kj}-\sum_s C^i_{js}R^s_{kh},$$

it can be seen that $\tilde{R}_j{}^i{}_{kh}$ is similar with $S_j{}^i{}_{kh}$ but $\dfrac{\partial}{y^k}$ is replaced by $\dfrac{\delta}{x^k}\cdot$ Thus its form is easily found from (3.4),

(3.8)
$$\tilde{R}_j{}^i{}_{kh}=0 \text{ for } i\neq j\neq k\neq h\neq i,\ \tilde{R}_i{}^i{}_{kh}=0 \text{ for } k\neq i, h\neq i,\ \tilde{R}_i{}^i{}_{ih}=0 \text{ for } i\neq h,$$
$$\tilde{R}_j{}^i{}_{ih}=\frac{1}{4\rho_i}\left[2\frac{\delta^2\rho_j}{\delta x^j\delta x^h}-\frac{1}{\rho_i}\frac{\delta\rho_i}{\delta x^j}\frac{\delta\rho_i}{\delta x^h}-\frac{1}{\rho_j}\frac{\delta\rho_j}{\delta x^h}\frac{\delta\rho_i}{\delta x^j}-\frac{1}{\rho_h}\frac{\delta\rho_h}{\delta x^j}\frac{\delta\rho_i}{\delta x^h}\right],$$
$$\tilde{R}_j{}^i{}_{kj}=\frac{1}{4\rho_i}\left[2\frac{\delta^2\rho_j}{\delta x^i\delta x^k}-\frac{2}{\rho_i}\frac{\delta\rho_i}{\delta x^k}\frac{\delta\rho_j}{\delta x^i}-\frac{1}{\rho_j}\frac{\delta\rho_j}{\delta x^k}\frac{\delta\rho_j}{\delta x^i}\right]\quad \text{for } i\neq j\neq k\neq i,$$
$$\tilde{R}_j{}^i{}_{ij}=\frac{1}{2\rho_i}\frac{\delta^2\rho_i}{(\delta x^k)^2}-\frac{1}{2\rho_i^2}\left(\frac{\delta\rho_i}{\delta x^j}\right)^2-\frac{1}{2\rho_i^2}\frac{\delta\rho_i}{\delta x^i}\frac{\delta\rho_j}{\delta x^i}+\frac{1}{2\rho_i}\frac{\delta^2\rho_j}{(\delta x^i)^2}-$$
$$-\frac{1}{4\rho_i\rho_j}\left(\frac{\delta\rho_j}{\delta x^i}\right)^2+\frac{1}{4\rho_i^2}\left(\frac{\delta\rho_i}{\delta x^j}\right)^2-\frac{1}{4\rho_i\rho_j}\frac{\delta\rho_i}{\delta x^j}\frac{\delta\rho_j}{\delta x^j}+$$
$$+\frac{1}{4\rho_i^2}\frac{\delta\rho_j}{\delta x^i}\frac{\delta\rho_i}{\delta x^i}+\sum_{j\neq s}\frac{1}{4\rho_i\rho_s}\frac{\delta\rho_j}{\delta x^s}\frac{\delta\rho_i}{\delta x^s}\quad \text{for } i\neq j.$$

Next, in order to find ${R_j}^i{}_{kh}$ it suffices to find an explicit form for $\sum_s C^i_{js} R^s_{kh}$ denoted for brevity by ${A_j}^i{}_{kh}$. We have a first form of it as follows

$$
\begin{aligned}
&{A_j}^i{}_{kh} = C^i_{ji} R^i_{kh} + C^i_{jj} R^j_{kh} \text{ for } i \neq j \neq k \neq h \neq i,\\
&{A_i}^i{}_{kh} = \sum_{\substack{s\neq k,\, s\neq h\\ s\neq i}} C^i_{is} R^s_{kh} + C^i_{ik} R^k_{kh} - C^i_{ih} R^h_{hk} + C^i_{ii} R^i_{kh} \text{ for } i \neq k \neq h \neq i,\\
&{A_i}^i{}_{ih} = \sum_{s\neq i,\, s\neq h} C^i_{is} R^s_{ih} + C^i_{ii} R^i_{ih} - C^i_{ih} R^h_{hi}, \qquad (3.9)\\
&{A_j}^i{}_{ih} = C^i_{jj} R^j_{ih} + C^i_{ij} R^i_{ih},\\
&{A_j}^i{}_{kj} = -C^i_{ij} R^i_{jk} - C^i_{jj} R^j_{jk},\\
&{A_j}^i{}_{ij} = C^i_{ij} R^i_{ij} + C^i_{jj} R^j_{ij}.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
R_{jk} &= -{R_j}^i{}_{ik} - \sum_{i\neq j,\, i\neq k} {R_j}^i{}_{ik} \quad \text{for } j \neq k;\\
R_{jj} &= -\sum_{i\neq j} {R_j}^i{}_{ij},
\end{aligned} \qquad (3.10)
$$

by (3.7)–(3.9) we obtain a first form of R_{jk}

$$
\begin{aligned}
R_{jk} =& \sum_{\substack{s\neq j\\ s\neq k}} C^j_{js} R^s_{jk} + C^j_{jj} R^j_{jk} - C^j_{jk} R^k_{ki} -\\
&-\sum_{\substack{i\neq j\\ i\neq k}} \frac{1}{4\rho_i}\left[2\frac{\delta^2 \rho_i}{\delta x^j \delta x^k} - \frac{1}{\rho_i}\frac{\delta\rho_i}{\delta x^k} - \frac{1}{\rho_j}\frac{\delta\rho_j}{\delta x^k}\frac{\delta\rho_i}{\delta x^j} - \frac{1}{\rho_k} x^j \frac{\delta\rho_i}{\delta x^k}\right] -\\
&-\sum_{\substack{i\neq j\\ i\neq k}} (C^i_{jj} R^j_{ik} + C^i_{ij} R^i_{ik}), \qquad\qquad \text{for } j \neq k\\
R_{jj} =& \sum_{i\neq j} (C^i_{ij} R^i_{ij} + C^i_{jj} R^j_{ij}) - \sum_{i\neq j} {\tilde{R}_j}{}^i{}_{ij}.
\end{aligned} \qquad (3.11)
$$

4 Concluding remarks

Our connection $C\Gamma(N)$ does not coincide to that which G.S. Asanov determined in [4]. As we have seen, the former has deflection, i.e., $D^i_j \neq 0$, while by the stipulation (21.3) in [4], the later is free of deflection. For a discussion at what extent the deflection tensor influences the existence of canonical metrical d-connection of a given GL-metric, we refer to [1].

The using of isotropic coordinates in writting of the static spherically symmetric metric (r_{ij}) allowed an easy extension of it to the Finslerian setting and simplified a little the calculation. We just can say that this fact made possible the computation of curvatures.

Our GL-metric (1.4)–(1.5) could be also considered as an extension of the Robertson–Walker metric to the Finslerian setting. It could provide various Finslerian Universes. However, the notion of Finslerian Universe has not yet a clear physical meaning and the computations seem hopeless. An algebraic manipulation programme would be helpful. Our paper could be a first step towards such a programme.

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