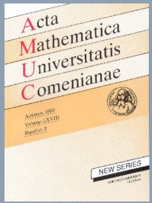

UNIFORM BOUNDEDNESS PRINCIPLE FOR UNBOUNDED OPERATORS

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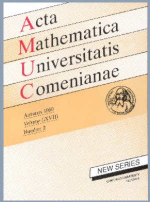
Abstract. A uniform boundedness principle for unbounded operators is derived. A particular case is: Suppose $\{T_i\}_{i\in I}$ is a family of linear mappings of a Banach space X into a normed space Y such that $\{T_i x : i \in I\}$ is bounded for each $x \in X$; then there exists a dense subset A of the open unit ball in X such that $\{T_i x : i \in I, x \in A\}$ is bounded. A closed graph theorem and a bounded inverse theorem are obtained for families of linear mappings as consequences of this principle. Some applications of this principle are also obtained.

1. Introduction

There are many forms for uniform boundedness principle. There is no known evidence for this principle for unbounded operators which generalizes classical uniform boundedness principle for bounded operators. The second section presents a uniform boundedness principle for unbounded operators. An application to derive Hellinger-Toeplitz theorem is also obtained in this section. A closed graph theorem and a bounded inverse theorem are obtained for families of linear mappings in the third section as consequences of this principle.

Let us assume the following: Every vector space X is over $\mathbb{R}$ or $\mathbb{C}$. An α-seminorm $(0 < \alpha \leq 1)$ is a mapping $p\colon X \to [0, \infty)$ such that $p(x + y) \leq p(x) + p(y)$, $p(ax) \leq |a|^\alpha p(x)$ for all $x, y \in X$

Received November 7, 2013.
2010 *Mathematics Subject Classification.* Primary 46A32, 47L60.
Key words and phrases. Uniform boundedness principle; closed graph theorem.



and for all scalars a. A subset M of a vector space X is said to be α-convex $(0 < \alpha \leq 1)$ if $a^{1/\alpha}x + (1-a)^{1/\alpha}y \in M$ for all $x, y \in M$ and for all scalars $0 \leq a \leq 1$ (see: [2]).

Every topological vector space is Hausdorff. An F-space is a complete metrizable topological vector space. A Frechet space is a locally convex F-space (see: [3]). The notations $\operatorname{int} A$ and $\operatorname{cl} A$ will denote interior and closure of a set A respectively.

2. Uniform Boundedness Principle

Theorem 2.1 (Uniform Boundednss Principle)**.** *Let X be an F-space, Y be a vector space and p be an α-seminorm on Y $(0 < \alpha \leq 1)$. Let $(T_i)_{i \in I}$ be a family of linear mappings from X into Y. Suppose $\{p(T_i x) : i \in I\}$ is bounded for each $x \in X$. Then there is an α-seminorm balanced open neighbourhood U of 0 in X, and there is a dense subset A of U such that $\{p(T_i x) : i \in I, x \in A\}$ is bounded and $\{ax : x \in A,\ a \text{ is a scalar}\}$ is a dense linear subspace of X.*

Proof. To each positive integer k, define a set $A_k = \{x \in X : p(T_i x) \leq k, \text{for all } i \in I\}$. As p is an α-seminorm, A_k is α-convex and balanced; so, $\operatorname{cl} A$ is α-convex and balanced. This implies that $\operatorname{int}\operatorname{cl} A_k$ is also α-convex and symmetric. Indeed, $\operatorname{int}\operatorname{cl} A_k = \operatorname{int}\operatorname{cl}(-A_k) = -\operatorname{int}\operatorname{cl} A_k$ and if $0 < a < 1$, then $a^{1/\alpha}\operatorname{int}\operatorname{cl} A_k + (1-a)^{1/\alpha}\operatorname{int}\operatorname{cl} A_k$ is open in X and it is contained in $\operatorname{cl} A_k$, and hence it is contained in $\operatorname{int}\operatorname{cl} A_k$. If $x \in \operatorname{int}\operatorname{cl} A_k \neq \emptyset$, then $0 = (1/2)^{1/\alpha}x + (1-1/2)^{1/\alpha}(-x) \in \operatorname{int}\operatorname{cl} A_k$, and so $\operatorname{int}\operatorname{cl} A_k$ is balanced [3, Theorem 1.13(e)]. Thus $\operatorname{int}\operatorname{cl} A_k$ is balanced if it is nonempty. Since $X = \cup_{j=1}^{\infty} A_k$, $\operatorname{int}\operatorname{cl} A_k \neq \emptyset$ for some k. Take $U = \operatorname{int}\operatorname{cl} A_k$ and $Z = \{ax : x \in A_k \cap U, a \text{ is a scalar}\}$. If $x, y \in Z$, there is a scalar $a > 0$ such that $ax, ay \in A_k \cap U$; this implies that $(1/2)^{1/\alpha}ax + (1-1/2)^{1/\alpha}ay \in A_k \cap U$, and hence $x + y \in Z$. This shows that Z is a linear subspace of X. If a nonzero $x \in X$ and a neighbourhood V of 0 such that $V \subset U$ are given, then let us find a scalar $a > 0$ such that $ax \in U$. We can find a balanced neighbourhood $W \subset V$ of 0 such that $ax + aW = a(x + W) \subset U \subset cl\ A_k$. So, there is an element $y \in W \subset V$ such that $ax + ay \in A_k \cap U$.

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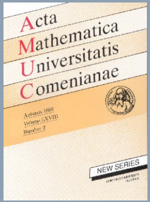

Thus $x + y \in Z \cap (x + V)$. This proves that Z is a dense linear subspace of X. Since $U \subset \operatorname{cl} A_k$ and U is open, $U \cap A_k$ is dense in U. So, we take $A = U \cap A_k$. This completes the proof. □

The arguments imply the following results.

Theorem 2.2. *Let X be an F-space, $\{Y_i\}_{i\in I}$ be a collection of vector spaces and p_i be an α-seminorm on Y_i for a fixed α, $(0 < \alpha \le 1)$. Let $\{T_i\}_{i\in I}$ be a family of linear mappings $T_i \colon X \to Y_i$. Suppose $\{p_i(T_i x) : i \in I\}$ is bounded for each $x \in X$. Then there is an α-convex balanced open neighbourhood U of 0 in X, and there is a dense subset A of U such that $\{p_i(T_i x) : i \in I, x \in A\}$ is bounded and $\{ax : x \in A, a \text{ is a scalar}\}$ is a dense linear subspace of X.*

Theorem 2.3. *Let X be an F-space, and Y be a topological vector space. Suppose $\{p_j\}_{j\in J}$ is a family that defines the topology of Y, where each p_j is an α_j-seminorm on Y, $(0 < \alpha_j \le 1)$. Let $\{T_i\}_{i\in I}$ be a family of linear mappings from X into Y such that $\{T_i x : i \in I\}$ is bounded for each $x \in X$. Then for each $j \in J$, there is an α_j-convex open neighbourhood U_j of 0 in X, and there is a dense subset A_j of U_j such that $\{p_j(T_i x) : i \in I, x \in A_j\}$ is bounded.*

Corollary 2.4. *Let X be a Banach space, Y be a normed space, and $\{T_i\}_{i\in I}$ be a family of linear mappings of X into Y. Suppose $\{T_i x : i \in I\}$ is bounded for each $x \in X$. Then there exists a dense subset A of the unit open ball in X such that $\{T_i x : i \in I, x \in A\}$ is bounded.*

The following theorem is an application of our principle.

Theorem 2.5. *Suppose $\{T_i\}_{i\in I}$ and $\{T_i^*\}_{i\in I}$ are two families of linear mappings from X into Y and from Y' into X', respectively, when X and Y are normed spaces, and when X' and Y' are respective duals, and suppose they satisfy $(f \circ T_i)x = (T_i^* f)(x)$ for all $f \in Y'$, $x \in X$, $i \in I$. Suppose $\{T_i^* f : i \in I\}$ is bounded for each $f \in Y'$. Then $\{T_i\}_{i\in I}$ and $\{T_i^*\}_{i\in I}$ are equicontinuous on X and Y', respectively.*

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Proof. By our principle, there is a convex open neighbourhood U of 0 in Y' and there is a dense subset A of U such that $\{T_i^* f : i \in I,\ f \in A\}$ is bounded. We lose no generality by assuming that U is the open unit ball in Y'. There is a $k > 0$ such that $\|T_i^* f\| \le k$ for all $i \in I$, for all $f \in A$. Then

$$|(f \circ T_i)x| = |(T_i^* f)(x)| \le \|T_i^* f\|\|x\| \le k\|x\| \qquad \text{for every } f \in A,\ i \in I,\ x \in X$$

That is

$$\sup_{f \in A} |f(T_i x)| \le k\|x\| \qquad \text{for every } x \in X,\ i \in I.$$

This implies that

$$\sup_{f \in U} |f(T_i x)| \le k\|x\| \qquad \text{for every } x \in X,\ i \in I.$$

Therefore

$$\|T_i x\| = \sup_{\|f\| \le 1} |f(T_i x)| \le k\|x\| \qquad \text{for every } x \in X,\ i \in I$$

Thus $(T_i)_{i \in I}$ is equicontinuous on X. Note that $\|T_i\| = \|T_i^*\|$ for all $i \in I$ (see [1]). So, $(T_i^*)_{i \in I}$ is also equicontinuous. □

Remark 2.6. If I is finite, then $\{T_i^*(f) : i \in I\}$ is bounded for every $f \in Y'$.

The argument used in the proof of Theorem 2.5 implies the next theorem

Theorem 2.7 (A General form of Hellinger-Toeplitz theorem). *Suppose $\{T_i\}_{i \in I}$ and $\{T_i^*\}_{i \in I}$ are two families of linear mappings from a Hilbert space H_1 into a Hilbert space H_2 and they satisfy $\langle T_i x, y\rangle = \langle x, T_i^* y\rangle$ for every $i \in I$, $x \in H_1$, $y \in H_2$. Suppose $\{T_i^* y : i \in I\}$ is bounded for every $y \in H_2$ or $\{T_i x : i \in I\}$ is bounded for every $x \in H_1$. Then $\{T_i\}_{i \in I}$ is equicontinuous on H_1 and $\{T_i^*\}_{i \in I}$ is equicontinuous on H_2.*

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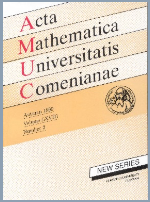

3. A Closed Graph Theorem

Theorem 3.1 (Closed Graph Theorem). *Let $\{T_i\}_{i \in I}$ be a family of linear mappings from an F-space X into an F-space Y. Suppose the topology of Y is defined by a countable collection $\{p_n\}_{n=1}^{\infty}$, where each p_n is an α_n-seminorm, $(0 < \alpha_n \leq 1)$. Suppose each T_i has closed graph in $X \times Y$. Assume further that $\{T_i x : i \in I\}$ is bounded for each $x \in X$. Then $\{T_i\}_{i \in I}$ is equicontinuous on X.*

Proof. By a classical closed graph theorem, each T_i is continuous. By Theorem 2.3, for each given $n = 1, 2, \dots,$ there exists an open neighbourhood U_n of 0 in X and there is a dense subset A_n of U_n such that $p_n(T_i x) \leq k_n$ for all $i \in I$, for some $k_n > 0$; for all $i \in I$ and for all $x \in A_n$. Hence $p_n(T_i x) \leq k_n$ for all $i \in I$, $x \in U_n$, because A_n is dense in U_n and each T_i is continuous. This completes the proof. □

Note that a classical Uniform boundedness principle gives this result. However, this proof gives the argument to derive the classical uniform boundedness principle for bounded operators from our principle for unbounded operators. One may also obtain a proof for the following theorem by using a classical result.

Theorem 3.2 (Bounded inverse theorem). *Let $(T_i)_{i \in I}$ be a family of injective continuous linear mappings from an F-space X onto an F-space Y. Suppose the topology on X is defined by a countable collection (p_n), where each p_n is an α_n-seminorm. Suppose $\{T_i^{-1} y : i \in I\}$ is bounded in X for each $y \in Y$. Then $(T_i)_{i \in I}$ is equicontinuous on Y.*

Proof. Each T_i^{-1} has a closed graph in $Y \times X$. Apply the previous closed graph theorem to obtain this result. □

In the following variant of the uniform boundedness principle, the Baire category theorem is applied to a locally compact set rather than to a complete metric space.

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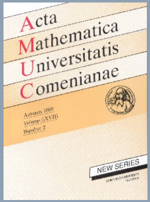

Let X and Y be two topological vector spaces, p be an α-seminorm of Y and K be a locally compact α-convex subset of X. Let $\{T_i\}_{i\in I}$ be a family of linear mappings from X into Y. Suppose $\{p(T_i x) : i \in I\}$ is bounded for each $x \in K$. Then there is a nonempty α-convex open set U relative to K and there is a dense subset A of U such that $\{p(T_i x) : i \in I,\ x \in A\}$ is bounded.

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