The size of triangulations supporting a given link

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Abstract

Let $T$ be a triangulation of $S^3$ containing a link $L$ in its 1-skeleton. We give an explicit lower bound for the number of tetrahedra of $T$ in terms of the bridge number of $L$. Our proof is based on the theory of almost normal surfaces.

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1 Introduction

In this paper, we prove the following result.

**Theorem 1** Let \( L \subset S^3 \) be a tame link with bridge number \( b(L) \). Let \( T \) be a triangulation of \( S^3 \) with \( n \) tetrahedra such that \( L \) is contained in the 1–skeleton of \( T \). Then

\[
 n > \frac{1}{14} \sqrt{\log_2 b(L)},
\]

or equivalently

\[
 b(L) < 2^{196n^2}.
\]

The definition of the bridge number can be found, for instance, in [2]. So far as is known to the author, Theorem 1 gives the first estimate for \( n \) in terms of \( L \) that does not rely on additional geometric or combinatorial assumptions on \( T \).

We show in [13] that the bound for \( b(L) \) in Theorem 1 can not be replaced by a sub-exponential bound in \( n \). More precisely, there is a constant \( c \in \mathbb{R} \) such that for any \( i \in \mathbb{N} \) there is a triangulation \( T_i \) of \( S^3 \) with \( c \cdot i \) tetrahedra, containing a two-component link \( L_i \) in its 1–skeleton with \( b(L_i) > 2^{i-2} \).

The relationship of geometric and combinatorial properties of a triangulation of \( S^3 \) with the knots in its 1–skeleton has been studied earlier, see [6], [15], [1], [3], [7]. For any knot \( K \subset S^3 \) there is a triangulation of \( S^3 \) with \( k \) tetrahedra and let \( K \subset S^3 \) be a knot formed by a path of \( k \) edges. If \( T \) is shellable (see [3]) or the dual cellular decomposition is shellable (see [1]), then \( b(K) \leq \frac{1}{2}k \). If \( T \) is vertex decomposable then \( b(K) \leq \frac{1}{3}k \), see [3].

We reduce Theorem 1 to Theorem 2 below, for which we need some definitions. Denote \( I = [0, 1] \). Let \( M \) be a closed 3–manifold with a triangulation \( T \). The \( i \)–skeleton of \( T \) is denoted by \( T^i \). Let \( S \) be a surface and let \( H: S \times I \to M \) be an embedding, so that \( T^1 \subset H(S^2 \times I) \). A point \( x \in T^1 \) is a critical point of \( H \) if \( H_\xi = H(S \times \xi) \) is not transversal to \( T^1 \) in \( x \), for some \( \xi \in I \). We call \( H \) a \( T^1 \)–Morse embedding, if \( H \) is in general position with respect to \( T^1 \); we give a more precise definition in Section 5. Denote by \( c(H, T^1) \) the number of critical points of \( H \).

**Theorem 2** Let \( T \) be a triangulation of \( S^3 \) with \( n \) tetrahedra. There is a \( T^1 \)–Morse embedding \( H: S^2 \times I \to S^3 \) such that \( T^1 \subset H(S^2 \times I) \) and

\[
 c(H, T^1) < 2^{196n^2}.
\]
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For a link \( L \subset T^1 \), it is easy to see that \( b(L) \leq \frac{1}{2} \min_H \{ c(H, T^1) \} \), where the minimum is taken over all \( T^1 \)-Morse embeddings \( H : S^2 \times I \to S^3 \) with \( L \subset H(S^2 \times I) \). Thus Theorem 1 is a corollary of Theorem 2.

Our proof of Theorem 2 is based on the theory of almost 2-normal surfaces. Kneser [14] introduced 1-normal surfaces in his study of connected sums of 3-manifolds. The theory of 1-normal surfaces was further developed by Haken (see [8], [9]). It led to a classification algorithm for knots and for sufficiently large 3-manifolds, see for instance [11], [17]. The more general notion of almost 2-normal surfaces is due to Rubinstein [19]. With this concept, Rubinstein and Thompson found a recognition algorithm for \( S^3 \), see [19], [22], [16]. Based on the results discussed in a preliminary version of this paper [12], the author [13] and Mijatović [18] independently obtained a recognition algorithm for \( S^3 \) using local transformations of triangulations.

We outline here the proof of Theorem 2. Let \( T \) be a triangulation of \( S^3 \) with \( n \) tetrahedra. If \( S \subset S^3 \) is an embedded surface and \( S \cap T^1 \) is finite, then set \( \|S\| = \text{card}(S \cap T^1) \). Let \( S_1, \ldots, S_k \subset S^3 \) be surfaces. A surface that is obtained by joining \( S_1, \ldots, S_k \) with some small tubes in \( M \setminus T^1 \) is called a tube sum of \( S_1, \ldots, S_k \).

Based on the Rubinstein–Thompson algorithm, we construct a system \( \bar{\Sigma} \subset S^3 \) of pairwise disjoint 2-normal 2-spheres such that \( \|\bar{\Sigma}\| \) is bounded in terms of \( n \) and any 1-normal sphere in \( S^3 \setminus \bar{\Sigma} \) is parallel to a connected component of \( \bar{\Sigma} \). The bound for \( \|\bar{\Sigma}\| \) can be seen as part of a complexity analysis for the Rubinstein–Thompson algorithm and relies on results on integer programming.

A \( T^1 \)-Morse embedding \( H \) then is constructed “piecewise” in the connected components of \( S^3 \setminus \bar{\Sigma} \), which means the following. There are numbers \( 0 < \xi_1 < \cdots < \xi_m < 1 \) such that:

1. \( \|H_0\| = \|H_1\| = 0 \).
2. There is one critical value of \( H \) on \( [0, \xi_1] \), corresponding to a vertex \( x_0 \in T^0 \).
3. The set of critical points of \( H \) on \( [\xi_m, 1] \) is \( T^0 \setminus \{x_0\} \).
4. For any \( i = 1, \ldots, m \), the sphere \( H_{\xi_i} \) is a tube sum of components of \( \bar{\Sigma} \).
5. The critical points of \( H \) on \( [\xi_i, \xi_{i+1}] \) are contained in a single connected component \( N_i \) of \( S^3 \setminus \bar{\Sigma} \).
6. The function \( \xi \mapsto \|H_\xi\| \) is monotone in any interval \( [\xi_i, \xi_{i+1}] \), for \( i = 1, \ldots, m-1 \).

This is depicted in Figure 1, where the components of \( \bar{\Sigma} \) are dotted. The components \( N_i \) run over all components of \( S^3 \setminus \bar{\Sigma} \) that are not regular neighbourhoods of vertices of \( T \). Thus an estimate for \( m \) is obtained by an estimate
for the number of components of $\Sigma$. By monotonicity of $\|H_\xi\|$, the number of critical points in $N_i$ is bounded by $\frac{1}{2}\|\partial N_i\| \leq \frac{1}{2}\|\Sigma\|$. This together with the bound for $m$ yields the claimed estimate for $c(H,T^1)$.

The main difficulty in constructing $H$ is to assure property (5). For this, we introduce the notions of upper and lower reductions. If $S'$ is an upper (resp. lower) reduction of a surfaces $S \subset S^3$, then $S$ is isotopic to $S'$ such that $\|\cdot\|$ is monotonely non-increasing under the isotopy. Let $N$ be a connected component of $S^3 \setminus \Sigma$ with $\partial N = S_0 \cup S_1 \cup \cdots \cup S_k$. We show that there is a tube sum $S$ of $S_1, \ldots, S_k$ such that either $S$ is a lower reduction of $S_0$, or $S_0$ is an upper reduction of $S$. Finally, if $H_{\xi_i}$ is a tube sum of $S_0$ with some surface $S' \subset S^3 \setminus N$, then $H_{[\xi_i, \xi_{i+1}]}$ is induced by the lower reductions (resp. the inverse of the upper reductions) relating $S_0$ with $S$. Then $H_{\xi_{i+1}}$ is a tube sum of $S$ with $S'$, assuring properties (3)--(5).

The paper is organized as follows. In Section 2, we recall basic properties of $k$--normal surfaces. It is well known that the set of 1--normal surfaces in a triangulated 3--manifold is additively generated by so-called fundamental surfaces. In Section 3, we generalize this to 2--normal surfaces contained in sub-manifolds of triangulated 3--manifolds. The system $\Sigma$ of 2--normal spheres is constructed in Section 4, in the more general setting of closed orientable 3--manifolds. In Section 5, we recall the notions of almost $k$--normal surfaces (see [16]) and of impermeable surfaces (see [22]), and introduce the new notion of split equivalence. We discuss the close relationship of almost 2--normal surfaces and impermeable surfaces. This relationship is well known (see [22], [16]), but the proofs are only partly available. For completeness we give a proof in the last Section 9. In Section 6 we exhibit some useful properties of almost 1--normal surfaces. The notions of upper and lower reductions are introduced in Section 7. The proof of Theorem 2 is finished in Section 8.

In this paper, we denote by $\#(X)$ the number of connected components of a topological space $X$. If $X$ is a tame subset of a 3--manifold $M$, then $U(X) \subset M$.
denotes a regular neighbourhood of $X$ in $M$. For a triangulation $T$ of $M$, the number of its tetrahedra is denoted by $t(T)$.

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## 2 A survey of $k$–normal surfaces

Let $M$ be a closed 3–manifold with a triangulation $T$. The number of its tetrahedra is denoted by $t(T)$. An isotopy mod $T^n$ is an ambient isotopy of $M$ that fixes any simplex of $T^n$ set-wise. Some authors call an isotopy mod $T^2$ a normal isotopy.

**Definition 1** Let $\sigma$ be a 2–simplex and let $\gamma \subset \sigma$ be a closed embedded arc with $\gamma \cap \partial \sigma = \partial \gamma$, disjoint to the vertices of $\sigma$. If $\gamma$ connects two different edges of $\sigma$ then $\gamma$ is called a normal arc. Otherwise, $\gamma$ is called a return.

We denote the number of connected components of a topological space $X$ by $\#(X)$. Let $\sigma$ be a 2–simplex with edges $e_1,e_2,e_3$. If $\Gamma \subset \sigma$ is a system of normal arcs, then $\Gamma$ is determined by $\Gamma \cap \partial \sigma$, up to isotopy constant on $\partial \sigma$, and $e_1$ is connected with $e_2$ by $\frac{1}{2}(\#(\Gamma \cap e_1) + \#(\Gamma \cap e_2) - \#(\Gamma \cap e_3))$ arcs in $\Gamma$.

**Definition 2** Let $S \subset M$ be a closed embedded surface transversal to $T^2$. We call $S$ pre-normal, if $S \setminus T^2$ is a disjoint union of discs and $S \cap T^2$ is a union of normal arcs in the 2–simplices of $T$.

The set $S \cap T^1$ determines the normal arcs of $S \cap T^2$. For any tetrahedron $t$ of $T$, the components of $S \cap t$, being discs, are determined by $S \cap \partial t$, up to isotopy fixed on $\partial t$. Thus we obtain the following lemma.

**Lemma 1** A pre-normal surface $S \subset M$ is determined by $S \cap T^1$, up to isotopy mod $T^2$. \hfill \Box

**Definition 3** Let $S \subset M$ be a pre-normal surface and let $k$ be a natural number. If for any connected component $C$ of $S \setminus T^2$ and any edge $e$ of $T$ holds $\#(\partial C \cap e) \leq k$, then $S$ is $k$–normal.
We are mostly interested in 1– and 2–normal surfaces. Let $S$ be a 2–normal surface and let $t$ be a tetrahedron of $T$. Then the components of $S \cap t$ are copies of triangles, squares and octagons, as in Figure 2. For any tetrahedron $t$, there are 10 possible types of components of $S \cap t$: four triangles (one for each vertex of $t$), three squares (one for each pair of opposite edges of $t$), and three octagons. Thus there are $10t(T)$ possible types of components of $S \setminus T^2$. Up to isotopy mod $T^2$, the set $S \setminus T^2$ is described by the vector $\gamma(S)$ of $10t(T)$ non-negative integers that indicates the number of copies of the different types of discs occuring in $S \setminus T^2$. Note that the 1–normal surfaces are formed by triangles and squares only.

We will describe the non-negative integer vectors that represent 2–normal surfaces. Let $S \subset M$ be a 2–normal surface and let $x_{t,1}, \ldots, x_{t,6}$ be the components of $\gamma(S)$ that correspond to the squares and octagons in some tetrahedron $t$. It is impossible that in $S \cap t$ occur two different types of squares or octagons, since two different squares or octagons would yield a self-intersection of $S$. Thus all but at most one of $x_{t,1}, \ldots, x_{t,6}$ vanish for any $t$. This property of $\gamma(S)$ is called compatibility condition.

Let $\gamma$ be a normal arc in a 2–simplex $\sigma$ of $T$ and $t_1, t_2$ be the two tetrahedra that meet at $\sigma$. In both $t_1$ and $t_2$ there are one triangle, one square and two octagons that contain a copy of $\gamma$ in its boundary. Moreover, each of them contains exactly one copy of $\gamma$. Let $x_{t_i,1}, \ldots, x_{t_i,4}$ be the components of $\gamma(S)$ that correspond to these types of discs in $t_i$, where $i = 1, 2$. Since $\partial S = \emptyset$, the number of components of $S \cap t_1$ containing a copy of $\gamma$ equals the number of components of $S \cap t_2$ containing a copy of $\gamma$. That is to say $x_{t_1,1} + \cdots + x_{t_1,4} = x_{t_2,1} + \cdots + x_{t_2,4}$. Thus $\gamma(S)$ satisfies a system of linear Diophantine equations, with one equation for each type of normal arcs. This property of $\gamma(S)$ is called matching condition. The next claim states that the compatibility and the matching conditions characterize the vectors that represent 2–normal surfaces. A proof can be found in [11], Chapter 9.
Proposition 1 Let $\mathbf{x}$ be a vector of $10 t(T)$ non-negative integers that satisfies both the compatibility and the matching conditions. Then there is a 2-normal surface $S \subset M$ with $\tri(S) = \mathbf{x}$. □

Two 2-normal surfaces $S_1, S_2$ are called compatible if the vector $\tri(S_1) + \tri(S_2)$ satisfies the compatibility condition. It always satisfies the matching condition. Thus if $S_1$ and $S_2$ are compatible, then there is a 2-normal surface $S$ with $\tri(S) = \tri(S_1) + \tri(S_2)$, and we denote $S = S_1 + S_2$. Conversely, let $S$ be a 2-normal surface, and assume that there are non-negative integer vectors $\mathbf{r}_1, \mathbf{r}_2$ that both satisfy the matching condition, with $\tri(S) = \mathbf{r}_1 + \mathbf{r}_2$. Then both $S_1$ and $S_2$ satisfy the compatibility condition. Thus there are 2-normal surfaces $S_1, S_2$ with $S = S_1 + S_2$. The Euler characteristic is additive, i.e., $\chi(S_1 + S_2) = \chi(S_1) + \chi(S_2)$, see [11].

Remark 1 The addition of 2-normal surfaces extends to an addition on the set of pre-normal surfaces as follows. If $S_1, S_2 \subset M$ are pre-normal surfaces, then $S_1 + S_2$ is the pre-normal surface that is determined by $T^1 \cap (S_1 \cup S_2)$. The addition yields a semi-group structure on the set of pre-normal surfaces. This semi-group is isomorphic to the semi-group of integer points in a certain rational convex cone that is associated to $T$. The Euler characteristic is not additive with respect to the addition of pre-normal surfaces.

3 Fundamental surfaces

We use the notations of the previous section. The power of the theory of 2-normal surfaces is based on the following two finiteness results.

Proposition 2 Let $S \subset M$ be a 2-normal surface comprising more than $10 t(T)$ two-sided connected components. Then two connected components of $S$ are isotopic mod $T^2$. □

This is proven in [9], Lemma 4, for 1-normal surfaces. The proof easily extends to 2-normal surfaces.

Theorem 3 Let $N \subset M \setminus U(T^0)$ be a sub-3-manifold whose boundary is a 1-normal surface. There is a system $F_1, \ldots, F_q \subset N$ of 2-normal surfaces such that

$$\|F_i\| < \|\partial N\| \cdot 2^{18 t(T)}$$

for $i = 1, \ldots, q$, and any 2-normal surface $F \subset N$ can be written as a sum $F = \sum_{i=1}^{q} k_i F_i$ with non-negative integers $k_1, \ldots, k_q$. 

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The surfaces $F_1, \ldots, F_q$ are called fundamental. Theorem 3 is a generalization of a result of [10] that concerns the case $N = M \setminus U(T^0)$.

The rest of this section is devoted to the proof of Theorem 3. The idea is to define a system of linear Diophantine equations (matching equations) whose non-negative solutions correspond to 2-normal surfaces in $N$. The fundamental surfaces $F_1, \ldots, F_q$ correspond to the Hilbert base vectors of the equation system, and the bound for $\|F_i\|$ is a consequence of estimates for the norm of Hilbert base vectors. Note that in an earlier version of this paper [12], we proved Theorem 3 in essentially the same way, but using handle decompositions of 3-manifolds rather than triangulations.

**Definition 4** A region of $N$ is a component $R$ of $N \cap t$, for a closed tetrahedron $t$ of $T$. If $\partial R \cap \partial N$ consists of two copies of one normal triangle or normal square then $R$ is a parallelity region.

**Definition 5** The class of a normal triangle, square or octagon in $N$ is its equivalence class with respect to isotopies mod $T^2$ with support in $U(N)$.

Let $t$ be a closed tetrahedron of $T$, and let $R \subset t$ be a region of $N$. One verifies that if $R$ is not a parallelity region then $\partial R \cap \partial N$ either consists of four normal triangles (“type I”) or of two normal triangles and one normal square (“type II”). If $R$ is of type I, then $R$ is isotopic mod $T^2$ to $t \setminus U(T^0)$, and any other region of $N$ in $t$ is a parallelity region. As in the previous section, $R$ contains four classes of normal triangles, three classes of normal squares and three classes of normal octagons. If $R$ is of type II, then $t$ contains at most one other region of $N$ that is not a parallelity region, that is then also of type II. A normal square or octagon in $t$ that is not isotopic mod $T^2$ to a component of $\partial R \cap \partial N$ intersects $\partial R$. Thus $R$ contains two classes of normal triangles and one class of normal squares.

Let $m(N)$ be the number of classes of normal triangles, squares and octagons in regions of $N$ of types I and II. If $N$ has $k$ regions of type I, then $N$ has $\leq 2(t(T) - k)$ regions of type II, thus $m(N) \leq 10k + 6(t(T) - k) \leq 10t(T)$. Let $\overline{m}(N)$ be the number of parallelity regions of $N$. It is easy to see that $\overline{m}(N) \leq \frac{1}{7} \#(\partial N \setminus T^2) \leq \frac{1}{7} \|\partial N\| \cdot t(T)$.

Any 2-normal surface $F \subset N$ is determined up to isotopy mod $T^2$ with support in $U(N)$ by the vector $\overline{m}(N)$ of $m(N) + \overline{m}(N)$ non-negative integers that count the number of components of $F \setminus T^2$ in each class of normal triangles, squares and octagons. Let $\gamma_1, \gamma_2 \subset T^2$ be normal arcs, and let $R_1, R_2$ be two regions of $N$ with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. For $i = 1, 2$, let $x_{i,1}, \ldots, x_{i,m_i}$ be the
components of $\mathcal{F}_N(F)$ that correspond to classes of normal triangles, squares and octagons in $R_i$ that contain $\gamma_i$ in its boundary. If $x_{1,1} + \cdots + x_{1,m_1} = x_{2,1} + \cdots + x_{2,m_2}$ then we say that $\mathcal{F}_N(F)$ satisfies the \textit{matching equation} associated to $(\gamma_1, R_1; \gamma_2, R_2)$.

For $i = 1, 2$, $R_i$ contains one class of normal triangles that contain a copy of $\gamma_i$ in its boundary. If $R_i$ is not a parallelity region, then $R_i$ contains one class of normal squares that contain a copy of $\gamma_i$ in its boundary. If $K_i$ is of type I, then $K_i$ additionally contains two classes of normal octagons containing a copy of $\gamma_i$ in its boundary. Thus if $R_i$ is a parallelity region then $m_i = 1$, if it is of type I then $m_i = 4$, and if it is of type II then $m_i = 2$.

For any 2–normal surface $F \subset N$, let $\xi_N(F) \in \mathbb{Z}_{\geq 0}^{m(N)}$ be the vector that collects the components of $\mathcal{F}_N(F)$ corresponding to the classes of normal triangles, squares and octagons in regions of $N$ of types I and II. As in the previous section, the vector $\xi_N(F)$ (resp. $\mathcal{F}_N(F)$) satisfies a \textit{compatibility condition}, i.e., for any region $R$ of $N$ vanish all but at most one components of $\xi_N(F)$ (resp. $\mathcal{F}_N(F)$) corresponding to classes of squares and octagons in $R$.

\textbf{Lemma 2} Suppose that any component of $N$ contains a region that is not a parallelity region. There is a system of matching equations concerning only regions of $N$ of types I and II, such that a vector $\xi \in \mathbb{Z}_{\geq 0}^{m(N)}$ satisfies these equations and the compatibility condition if and only if there is a 2–normal surface $F \subset N$ with $\xi_N(F) = \xi$. The surface $F$ is determined by $\xi_N(F)$, up to isotopy in $N \bmod T^2$.

\textbf{Proof} Let $\gamma \subset N \cap T^2$ be a normal arc. Let $R_1, R_2$ be the two regions of $N$ that contain $\gamma$. Let $F \subset N$ be a 2–normal surface. Since $\partial F = \emptyset$, the number of components of $F \cap R_1$ containing $\gamma$ and the number of components of $F \cap R_2$ containing $\gamma$ coincide. Thus $\mathcal{F}_N(F)$ satisfies the matching equation associated to $(\gamma, R_1; \gamma, R_2)$. We refer to these equations as $N$–matching equations. We will transform the system of $N$–matching equations by eliminating the components of $\mathcal{F}_N(F)$ that do not belong to $\xi_N(F)$.

Let $\gamma_1, \gamma_2 \subset T^2$ be normal arcs, and let $R_1, R_2$ be two different regions of $N$ with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. Assume that $R_1$ is a parallelity region of $N$. Then $m_1 = 1$, thus the matching equation associated to $(\gamma_1, R_1; \gamma_2, R_2)$ is of the form $x_{1,1} = x_{2,1} + \cdots + x_{2,m_2}$. Hence we can eliminate $x_{1,1}$ in the $N$–matching equations. For any region $R_3$ of $N$ and any normal arc $\gamma_3 \subset \partial R_3$, the elimination transforms the matching equation associated to $(\gamma_1, R_1; \gamma_3, R_3)$ into the matching equation associated to $(\gamma_2, R_2; \gamma_3, R_3)$. We iterate the elimination process. Since any component of $N$ contains a region that is not a
parallelity region, we eventually transform the system of $N$–matching equations to a system $\mathfrak{A}$ of matching equations that concern only regions of $N$ of types $I$ and $II$.

Let $\mathbf{r} \in \mathbb{Z}^{m(N)}_{\geq 0}$ be a solution of $\mathfrak{A} \cdot \mathbf{r} = 0$. By the elimination process, there is a unique extension of $\mathbf{r}$ to a solution $\mathbf{r}'$ of the $N$–matching equations. If $\mathbf{r}'$ satisfies the compatibility condition then so does $\mathbf{r}'$, since a parallelity region contains at most one class of normal squares. Now the lemma follows by Proposition 1, that is proven in [11].

**Proof of Theorem 3** It is easy to verify that if $R$ is a parallelity region then there is only one class of 2–normal pieces in $R$. If a component $N_1$ of $N$ is a union of parallelity regions, then $N_1$ is a regular neighbourhood of a 1–normal surface $F_1 \subset N_1$, that has a connected non-empty intersection with each region of $N_1$. Any pre-normal surface in $N_1$ is a multiple of $F_1$ (thus, is 1-normal), see [8]. We have $\|F_1\| = \frac{1}{2} \|\partial N_1\|$. Thus by now we can suppose that any component of $N$ contains a region that is not a parallelity region.

By Lemma 2, the $\mathbf{r}$–vectors of 2–normal surfaces in $N$ satisfy a system of linear equations $\mathfrak{A} \cdot \mathbf{r} = 0$. By the following well known result on Integer Programming (see [21]), the non-negative integer solutions of such a system are additively generated by a finite set of solutions.

**Lemma 3** Let $\mathfrak{A} = (a_{ij})$ be an integer $(n \times m)$–matrix. Set

$$K = \left( \max_{i=1,\ldots,n} \sum_{j=1}^{m} a_{ij}^2 \right)^{1/2}.$$ 

There is a set $\{\mathbf{r}_1,\ldots,\mathbf{r}_p\}$ of non-negative integer vectors such that $\mathfrak{A} \cdot \mathbf{r}_i = 0$ for any $i = 1,\ldots,p$, the components of $\mathbf{r}_i$ are bounded from above by $mK^m$, and any non-negative integer solution $\mathbf{r}$ of $\mathfrak{A} \cdot \mathbf{r} = 0$ can be written as a sum $\mathbf{r} = \sum k_i \mathbf{r}_i$ with non-negative integers $k_1,\ldots,k_p$. 

The set $\{\mathbf{r}_1,\ldots,\mathbf{r}_p\}$ is called **Hilbert base** for $\mathfrak{A}$, if $p$ is minimal.

As in the previous section, if $F \subset N$ is a 2–normal surface and $\mathfrak{r}_N(F)$ is a sum of two non-negative integer solutions of $\mathfrak{A} \cdot \mathbf{r} = 0$ then there are 2–normal surfaces $F',F'' \subset N$ with $F = F' + F''$. Thus the surfaces $F_1,\ldots,F_q \subset N$ that correspond to Hilbert base vectors satisfying the compatibility condition additively generate the set of all 2–normal surfaces in $N$.

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It remains to bound \( \|F_i\| \), for \( i = 1, \ldots, q \). Since \( F_i \) is 2-normal and any edge of \( T \) is of degree \( \geq 3 \), we have \( \|F_i\| \leq \frac{8}{3} \#(F_i \setminus T^2) \). By the elimination process in the proof of Lemma 2, any component of \( \overline{r}_N(F_i) \) that corresponds to a parallelity region of \( N \) is a sum of at most four components of \( \overline{x}_N(F_i) \). By the bound for the components of \( \overline{x}_N(F_i) \) in Lemma 3 (with \( m = m(N) \) and \( K_2 = 8 \)) and our bounds for \( m(N) \) and \( m(N) \), we obtain

\[
\|F_i\| \leq \frac{8}{3} \left( m(N) + 4 \overline{m}(N) \right) \cdot \left( m(N) \cdot 2^{\frac{1}{2}m(N)} \right)
\]

\[
\leq \frac{8}{3} \left( 10 t(T) + \frac{2}{3} \|\partial N\| t(T) \right) \cdot 10 t(T) \cdot 2^{15 t(T)}
\]

\[
< (300 + 20 \|\partial N\|) \cdot t(T)^2 \cdot 2^{15 t(T)}.
\]

Using \( t(T) \geq 5 \) and \( \|\partial N\| > 0 \), we obtain \( \|F_i\| < \|\partial N\| \cdot 2^{18 t(T)} \).

\[ \square \]

4 Maximal systems of 1-normal spheres

Let \( T \) be a triangulation of a closed orientable 3-manifold \( M \). By Proposition 2, there is a system \( \Sigma \subset M \) of \( \leq 10 t(T) \) pairwise disjoint 1-normal spheres, such that any 1-normal sphere in \( M \setminus \Sigma \) is isotopic mod \( T^2 \) to a component of \( \Sigma \). We call such a system maximal. It is not obvious how to construct \( \Sigma \), in particular how to estimate \( \|\Sigma\| \) in terms of \( t(T) \). This section is devoted to a solution of this problem.

Construction 1 Set \( \Sigma_1 = \partial U(T^0) \) and \( N_1 = M \setminus U(T^0) \). Let \( i \geq 1 \). If there is a 1-normal fundamental projective plane \( P_i \subset N_i \) then set \( \Sigma_{i+1} = \Sigma_i \cup 2P_i \) and \( N_{i+1} = N_i \setminus U(P_i) \). Otherwise, if there is a 1-normal fundamental sphere \( S_i \subset N_i \) that is not isotopic mod \( T^2 \) to a component of \( \Sigma_i \), then set \( \Sigma_{i+1} = \Sigma_i \cup S_i \) and \( N_{i+1} = N_i \setminus U(S_i) \). Otherwise, set \( \Sigma = \Sigma_i \).

Since \( M \) is orientable, a projective plane \( P_i \) is one-sided and \( 2P_i \) is a sphere. By Proposition 2 and since embedded spheres are two-sided in \( M \), the iteration stops for some \( i < 10 t(T) \).

Lemma 4 \( \|\Sigma\| < 2^{185 t(T)^2} \).

Proof In Construction 1, we have

\[
\|\Sigma_{i+1}\| < \|\Sigma_i\| + 2\|\Sigma_i\| \cdot 2^{18 t(T)}
\]

\[
< \|\Sigma_i\| \cdot 2^{18 t(T) + 2}
\]

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by Theorem 3. The iteration stops after $< 10 t(T)$ steps, thus
\[ \|\Sigma\| < \|\Sigma_1\| \cdot 2^{180 t(T)^2 + 20 t(T)} \leq 2^{184 t(T)^2} , \]
using $t(T) \geq 5$. Since $\|\partial U(T^0)\|$ equals twice the number of edges of $T$, we have $\|\Sigma_1\| \leq 4 t(T)$, and the lemma follows.

Lemma 5  $\Sigma$ is maximal.

Proof  It is to show that any 1-normal sphere $S \subset M \setminus U(\Sigma)$ is isotopic mod $T^2$ to a component of $\Sigma$. Let $N$ be the component of $M \setminus U(\Sigma)$ that contains $S$. If $N$ contains a 1-normal fundamental projective plane $P$, then $N = U(P)$ by Construction 1. Thus $S = 2P = \partial N$, which is isotopic mod $T^2$ to a component of $\Sigma$. Hence we can assume that $N$ does not contain a 1-normal fundamental projective plane.

We express $S$ as a sum $\sum_{i=1}^q k_i F_i$ of fundamental surfaces in $N$. Since $\chi(S) = 2$ and the Euler characteristic is additive, one of the fundamental surfaces in the sum, say, $F_1$ with $k_1 > 0$, has positive Euler characteristic. It is not a projective plane by the preceding paragraph, thus it is a sphere. By construction of $\Sigma$, the sphere $F_1$ is isotopic mod $T^2$ to a component of $\Sigma$, thus it is parallel to a component of $\partial N$. Hence $F_1$ is disjoint to any 1-normal surface in $N$, up to isotopy mod $T^2$. Thus $S$ is the disjoint union of $k_1 F_1$ and $\sum_{i=2}^q k_i F_i$. Since $S$ is connected, it follows $S = F_1$. Thus $S$ is isotopic mod $T^2$ to a component of $\Sigma$.

We will extend $\Sigma$ to a system $\widetilde{\Sigma}$ of 2-normal spheres. To define $\widetilde{\Sigma}$, we need a lemma about 2-normal spheres in the complement of $\Sigma$.

Lemma 6  Let $N$ be a component of $M \setminus U(\Sigma)$. Assume that there is a 2-normal sphere in $N$ with exactly one octagon. Then there is a 2-normal fundamental sphere $F \subset N$ with exactly one octagon and $\|F\| < 2^{189 t(T)^2}$.

Proof  Let $S \subset N$ be a 2-normal sphere with exactly one octagon. If $N$ contains a 1-normal fundamental projective plane $P$, then $N = U(P)$ by Construction 1, and any pre-normal surface in $N$ is a multiple of $P$, i.e., is 1-normal. Thus since $S \subset N$ is not 1-normal, there is no 1-normal fundamental projective plane in $N$.

We write $S$ as a sum of 2-normal fundamental surfaces in $N$. Since $S$ has exactly one octagon, exactly one summand is not 1-normal. Since any projective plane in the sum is not 1-normal by the preceding paragraph, at most one
summand is a projective plane. Since $\chi(S) = 2$ and the Euler characteristic is additive, it follows that one of the fundamental surfaces in the sum is a sphere $F$.

Assume that $F$ is 1-normal, i.e., $S \neq F$. The construction of $\Sigma$ implies that $F$ is isotopic mod $T^2$ to a component of $\partial N$. Thus it is disjoint to any 2–normal surface in $N$. Therefore $S$ is a disjoint union of a multiple of $F$ and of a 2–normal surface with exactly one octagon, which is a contradiction since $S$ is connected. Hence $F$ contains the octagon of $S$. We have $\|F\| < \||\Sigma|| \cdot 2^{18t(T)}$ by Theorem 3. The claim follows with Lemma 4 and $t(T) \geq 5$.

The preceding lemma assures that the following construction works.

**Construction 2** For any connected component $N$ of $M \setminus U(\Sigma)$ that contains a 2–normal sphere with exactly one octagon, choose a 2–normal sphere $F_N \subset N$ with exactly one octagon and $\|F\| < 2^{189t(T)^2}$. Set

$$\bar{\Sigma} = \Sigma \cup \bigcup_N F_N.$$ 

Since $\#(\bar{\Sigma}) \leq 10t(T)$ by Proposition 2, it follows $\|\bar{\Sigma}\| < 10t(T) \cdot 2^{189t(T)^2} < 2^{190t(T)^2}$.

## 5 Almost $k$–normal surfaces and split equivalence

We shall need a generalization of the notion of $k$–normal surfaces. Let $M$ be a closed connected orientable 3–manifold with a triangulation $T$.

**Definition 6** A closed embedded surface $S \subset M$ transversal to $T^2$ is almost $k$–normal, if

1. $S \cap T^2$ is a union of normal arcs and of circles in $T^2 \setminus T^1$, and
2. for any tetrahedron $t$ of $T$, any edge $e$ of $t$ and any component $\zeta$ of $S \cap \partial t$ holds $\#(\zeta \cap e) \leq k$.

Our definition is similar to Matveev’s one in [16]. Note that there is a related but different definition of “almost normal” surfaces due to Rubinstein [19]. Any surface in $M$ disjoint to $T^1$ is almost 1–normal. Any almost $k$–normal surface that meets $T^1$ can be seen as a $k$–normal surface with several disjoint small tubes attached in $M \setminus T^1$, see [16]. The tubes can be nested. Of course there
are many ways to add tubes to a $k$–normal surface. We shall develop tools to
deal with this ambiguity.

Let $S \subset M$ be an almost $k$–normal surface. By definition, the connected
components of $S \cap T^2$ that meet $T^1$ are formed by normal arcs. Thus these
components define a pre-normal surface $S^\times$, that is obviously $k$–normal. It
is determined by $S \cap T^1$, according to Lemma 1. A disc $D \subset M \setminus T^1$ with
$\partial D \subset S$ is called a splitting disc for $S$. One obtains $S^\times$ by splitting $S$ along
splitting discs for $S$ that are disjoint to $T^2$, and isotopy mod $T^1$.

If two almost $k$–normal surfaces $S_1, S_2$ satisfy $S_1^\times = S_2^\times$, then $S_1$ and $S_2$
differ only by the choice of tubes. This gives rise to the following equivalence relation.

**Definition 7** Two embedded surfaces $S_1, S_2 \subset M$ transversal to $T^2$ are split
equivalent if $S_1 \cap T^1 = S_2 \cap T^1$ (up to isotopy mod $T^2$).

If two almost $k$–normal surfaces $S_1, S_2 \subset M$ are split equivalent, then $S_1^\times = S_2^\times$, up to isotopy mod $T^2$. In particular, two $k$–normal surfaces are split
equivalent if and only if they are isotopic mod $T^2$.

**Definition 8** If $S \subset M$ is an almost $k$–normal surface and $S^\times$ is the disjoint
union of $k$–normal surfaces $S_1, \ldots, S_n$, then we call $S$ a tube sum of $S_1, \ldots, S_n$.
We denote the set of all tube sums of $S_1, \ldots, S_n$ by $S_1 \circ \cdots \circ S_n$.

**Definition 9** Let $S = S_1 \cup \cdots \cup S_n \subset M$ be a surface transversal to $T^2$ with
$n$ connected components, and let $\Gamma \subset M \setminus T^1$ be a system of disjoint simple
arcs with $\Gamma \cap S = \partial \Gamma$. For any arc $\gamma$ in $\Gamma$, one component of $\partial U(\gamma) \setminus S$ is an
annulus $A_\gamma$. The surface

$$S^\Gamma = (S \setminus U(\Gamma)) \cup \bigcup_{\gamma \subset \Gamma} A_\gamma$$

is called the tube sum of $S_1, \ldots, S_n$ along $\Gamma$.

If $S_1, \ldots, S_n$ are $k$–normal, then $S^\Gamma \in S_1 \circ \cdots \circ S_n$.

We recall the concept of impermeable surfaces, that is central in the study of
almost 2–normal surfaces (see [22],[16]). Fix a vertex $x_0 \in T^0$. Let $S \subset M$ be
a connected embedded surface transversal to $T$. If $S$ splits $M$ into two pieces,
then let $B^+(S)$ denote the closure of the component of $M \setminus S$ that contains $x_0$,
and let $B^-(S)$ denote the closure of the other component. We do not include
$x_0$ in the notation “$B^+(S)$”, since in our applications the choice of $x_0$ plays no
essential role.
Definition 10  Let $S \subset M$ be a connected embedded surface transversal to $T^2$. Let $\alpha \subset T^1 \setminus T^0$ and $\beta \subset S$ be embedded arcs with $\partial \alpha = \partial \beta$. A closed embedded disc $D \subset M$ is a compressing disc for $S$ with string $\alpha$ and base $\beta$, if $\partial D = \alpha \cup \beta$ and $D \cap T^1 = \alpha$. If, moreover, $D \cap S = \beta$, then we call $D$ a bond of $S$.

Let $S \subset M$ be a connected embedded surface that splits $M$ and let $D$ be a compressing disc for $S$ with string $\alpha$. If the germ of $\alpha$ in $\partial \alpha$ is contained in $B^+(S)$ (resp. $B^-(S)$), then $D$ is upper (resp. lower). Let $D_1, D_2$ be upper and lower compressing discs for $S$ with strings $\alpha_1, \alpha_2$. If $D_1 \subset D_2$ or $D_2 \subset D_1$, then $D_1$ and $D_2$ are nested. If $D_1 \cap D_2 \subset \partial \alpha_1 \cap \partial \alpha_2$, then $D_1$ and $D_2$ are independent from each other.

Upper and lower compressing discs that are independent from each other meet in at most one point.

Definition 11  Let $S \subset M$ be a connected embedded surface that is transversal to $T^2$ and splits $M$. If $S$ has both upper and lower bonds, but no pair of nested or independent upper and lower compressing discs, then $S$ is impermeable.

Note that the impermeability of $S$ does not change under an isotopy of $S$ mod $T^1$. The next two claims state a close relationship between impermeable surfaces and (almost) 2-normal surfaces. By an octagon of an almost 2-normal surface $S \subset M$ in a tetrahedron $t$, we mean a circle in $S \cap \partial t$ formed by eight normal arcs. This corresponds to an octagon of $S^x$ in the sense of Figure 2.

Proposition 3  Any impermeable surface in $M$ is isotopic mod $T^1$ to an almost 2-normal surface with exactly one octagon.

Proposition 4  A connected 2-normal surface that splits $M$ and contains exactly one octagon is impermeable.

We shall need these statements later. As the author found only parts of the proofs in the literature (see [22],[16]), he includes proofs in Section 9.

We end this section with the definition of $T^1$-Morse embeddings and with the notion of thin position. Let $S$ be a closed 2-manifold and let $H: S \times I \to M$ be a tame embedding. For $\xi \in I$, set $H_\xi = H(S \times \xi)$.

Definition 12  An element $\xi \in I$ is a critical parameter of $H$ and a point $x \in H_\xi$ is a critical point of $H$ with respect to $T^1$, if $x$ is a vertex of $T$ or $x$ is a point of tangency of $H_\xi$ to $T^1$.
Definition 13 We call $H$ a $T^1$–Morse embedding, if it has finitely many critical parameters, to any critical parameter belongs exactly one critical point, and for any critical point $x \in T^1 \setminus T^0$ corresponding to a critical parameter $\xi$, one component of $U(x) \setminus H_\xi$ is disjoint to $T^1$. The number of critical points with respect to $T^1$ of a $T^1$–Morse embedding $H$ is denoted by $c(H, T^1)$.

The last condition in the definition of $T^1$–Morse embeddings means that any critical point of $H$ is a vertex of $T$ or a local maximum resp. minimum of an edge of $T$.

Definition 14 Let $F$ be a closed surface, let $J: F \times I \to M$ be a $T^1$–Morse embedding, and let $\xi_1, \ldots, \xi_r \in I$ be the critical parameters of $J$ with respect to $T^1$. The complexity $\kappa(J)$ of $J$ is defined as

$$\kappa(J) = \# \left( T^1 \setminus \left( \bigcup_{i=1}^{r} J_{\xi_i} \right) \right).$$

If $\kappa(J)$ is minimal among all $T^1$–Morse embeddings with the property $T^1 \subset J(F \times I)$, then $J$ is said to be in thin position with respect to $T^1$. This notion was introduced for foliations of 3-manifolds by Gabai [5], was applied by Thompson [22] for her recognition algorithm of $S^3$, and was also used in the study of Heegaard surfaces by Scharlemann and Thompson [20].

If $J(F \times \xi)$ splits $M$ and has a pair of nested or independent upper and lower compressing discs $D_1, D_2$, then an isotopy of $J$ along $D_1 \cup D_2$ decreases $\kappa(J)$, see [16], [22]. We obtain the following claim.

Lemma 7 Let $J: F \times I \to M$ be a $T^1$–Morse embedding in thin position and let $\xi \in I$ be a non-critical parameter of $J$. If $J(F \times \xi)$ has both upper and lower bonds, then $J(F \times \xi)$ is impermeable.

6 Compressing and splitting discs

Let $M$ be a closed connected 3-manifold with a triangulation $T$. In the lemmas that we prove in this section, we state technical conditions for the existence of compressing and splitting discs for a surface.

Lemma 8 Let $S_1, \ldots, S_n \subset M$ be embedded surfaces transversal to $T^2$ and let $S$ be the tube sum of $S_1, \ldots, S_n$ along a system $\Gamma \subset M \setminus T^1$ of arcs. Assume that $S$ splits $M$, and $\Gamma \subset B^-(S)$. If none of $S_1, \ldots, S_n$ has a lower compressing disc, then $S$ has no lower compressing disc.
Proof
Set $\Sigma = S_1 \cup \cdots \cup S_n$. Let $D \subset M$ be a lower compressing disc for $S$. One can assume that a collar of $\partial D \cap S$ in $D$ is contained in $\partial^-(S)$. Then, since by hypothesis $U(\Gamma) \cap \Sigma \subset \partial^-(S)$, any point in $\partial D \cap U(\Gamma) \cap \Sigma$ is endpoint of an arc in $D \cap \Sigma$. Therefore there is a sub-disc $D' \subset D$, bounded by parts of $\partial D$ and of arcs in $D \cap \Sigma$, that is a lower compressing disc for one of $S_1, \ldots, S_n$. □

Lemma 9 Let $S \subset M$ be a surface transversal to $T^2$ with upper and lower compressing discs $D_1, D_2$ such that $\partial (D_1 \cap D_2) \subset \partial D_2 \cap S$. Assume either that $(\partial D_1) \cap D_2 \subset T^1$ or that there is a splitting disc $D_m$ for $S$ such that $D_1 \cap D_m = \partial D_1 \cap \partial D_m = \{x\}$ is a single point and $D_2 \cap D_m = \emptyset$. Then $S$ has a pair of independent or nested upper and lower compressing discs.

Proof If $D_1 \cap D_2 \cap T^1$ comprises more than a single point then the string of $D_2$ is contained in the string of $D_1$. Thus $D_1 \cap S$ contains an arc different from the base of $D_1$, bounding in $D_1$ a lower compressing disc, that forms with $D_1$ a pair of nested upper and lower compressing discs for $S$.

Assume that a component $\gamma$ of $D_1 \cap D_2$ is a circle. Then there are discs $D'_1 \subset D_1$ and $D'_2 \subset D_2$ with $\partial D'_1 = \partial D'_2 = \gamma$. Since $\partial (D_1 \cap D_2) \subset \partial D_2$, $D'_2$ does not contain arcs of $D_1 \cap D_2$. Thus if we choose $\gamma$ innermost in $D_2$, then $D_1 \cap D'_2 = \gamma$. By cut-and-paste of $D_1$ along $D'_2$, one reduces the number of circle components in $D_1 \cap D_2$. Therefore we assume by now that $D_1 \cap D_2$ consists of isolated points in $\partial D_1 \cap \partial D_2$ and of arcs that do not meet $\partial D_1$.

Assume that there is a point $y \in (\partial D_1 \cap \partial D_2) \setminus T^1$. Then there is an arc $\gamma \subset \partial D_1$ with $\gamma = \{x, y\}$. Without assumption, let $\gamma \cap D_2 = \{y\}$. Let $A$ be the closure of the component of $U(\gamma) \setminus (D_1 \cup D_2 \cup D_m)$ whose boundary contains arcs in both $D_2$ and $D_m$. Define $D''_2 = ((D_2 \cup D_m) \setminus U(\gamma)) \cup A$, that is to say, $D''_2$ is the connected sum of $D_2$ and $D_m$ along $\gamma$. By construction, $(D_1 \cap D''_2) \setminus \partial D_1 = (D_1 \cap D_2) \setminus \partial D_1$, and $#(D_1 \cap D''_2) < #(D_1 \cap D_2)$. In that way, we remove all points of intersection of $(\partial D_1 \cap D''_2) \setminus T^1$. Thus by now we can assume that $D_1 \cap D_2$ consists of arcs in $D_1$ that do not meet $\partial D_1$ and possibly of a single point in $T^1$.

Let $\gamma \subset D_1 \cap D_2$ be an outermost arc in $D_2$, that is to say, $\gamma \cup \partial D_2$ bounds a disc $D' \subset D_2 \setminus T^1$ with $D_1 \cap D' = \gamma$. We move $D_1$ away from $D'$ by an isotopy mod $T^1$ and obtain a compressing disc $D'_1$ for $S$ with $D'_1 \cap D_2 = (D_1 \cap D_2) \setminus \gamma$. In that way, we remove all arcs of $D_1 \cap D_2$ and finally get a pair of independent upper and lower compressing discs for $S$. □

Lemma 10 Let $S \subset M$ be an almost 1-normal surface. If $S$ has a compressing disc, then $S$ is isotopic mod $T^1$ to an almost 1-normal surface with
a compressing disc contained in a single tetrahedron. In particular, $S$ is not 1–normal.

**Proof** Let $D$ be a compressing disc for $S$. Choose $S$ and $D$ up to isotopy of $S \cup D \mod T^1$ so that $S$ is almost 1–normal and $(D \cap T^2)$ is minimal. Choose an innermost component $\gamma \subset (D \cap T^2)$, which is possible as $D \cap T^2 \neq \emptyset$. There is a closed tetrahedron $t$ of $T$ and a component $C$ of $D \cap t$ that is a disc, such that $\gamma = C \cap \partial t$. Let $\sigma$ be the closed 2–simplex of $T$ that contains $\gamma$. We obtain three cases.

(1) Let $\gamma$ be a circle, thus $\partial C = \gamma$. Then there is a disc $D' \subset \sigma$ with $\partial D' = \gamma$ and a ball $B \subset t$ with $\partial B = C \cup D'$. By an isotopy mod $T^1$ with support in $U(B)$, we move $S \cup D$ away from $B$, obtaining a surface $S^*$ with a compressing disc $D^*$. If $S^*$ is almost 1–normal, then we obtain a contradiction to our choice as $(D^* \cap T^2) < (D \cap T^2)$.

(2) Let $\gamma$ be an arc with endpoints in a single component $c$ of $S \cap \sigma$. Since $S$ has no returns, $\gamma$ is not the string of $D$. We apply to $S \cup D$ an isotopy mod $T^1$ with support in $U(C)$ that moves $C$ into $U(C) \setminus t$, and obtain a surface $S^*$ with a compressing disc $D^*$. If $S^*$ is almost 1–normal, then we obtain a contradiction to our choice as $(D^* \cap T^2) < (D \cap T^2)$.

(3) Let $\gamma$ be an arc with endpoints in two different components $c_1, c_2$ of $S \cap \sigma$. If both $c_1$ and $c_2$ are normal arcs, then set $C' = C, c_1' = c_1$ and $c_2' = c_2$. If, say, $c_1$ is a circle, then we move $S \cup D$ away from $C$ by an isotopy mod $T^1$ with support in $U(C)$. If the resulting surface $S^*$ is still almost 1–normal, then we obtain a contradiction to the choice of $D$.

In either case, $S^*$ is not almost 1-normal, i.e., the isotopy introduces a return. Therefore there is a component of $C \setminus S$ with closure $C'$ such that $\partial C' \cap S$ connects two normal arcs $c_1', c_2'$ of $S \cap \sigma$.

Let $\gamma' = C' \cap \sigma$. Up to isotopy of $C'$ mod $T^2$ that is fixed on $\partial C' \cap S$, we assume that $\gamma' \cap (c_1' \cup c_2') \subset \partial \gamma'$. There is an arc $\alpha$ contained in an edge of $\sigma$ with $\partial \alpha \subset c_1' \cup c_2'$. For $i \in \{1, 2\}$, there is an arc $\beta_i \subset c_i'$ that connects $\alpha \cap c_i'$ with $\gamma' \cap c_i'$. The circle $\alpha \cup \beta_1 \cup \beta_2 \cup \gamma'$ bounds a closed disc $D' \subset \sigma$. Eventually $D' \cup C'$ is a compressing disc for $S$ contained in a single tetrahedron. \[\square\]

**Lemma 11** Let $S \subset M$ be a 1–normal surface and let $D$ be a splitting disc for $S$. Then, $(D, \partial D)$ is isotopic in $(M \setminus T^1, S \setminus T^1)$ to a disc embedded in $S$.

**Proof** We choose $D$ up to isotopy of $(D, \partial D)$ in $(M \setminus T^1, S \setminus T^1)$ so that $(\#((\partial D) \cap T^2), \#(D \cap T^2))$ is minimal in lexicographic order. Assume that
\[ \partial D \cap T^2 \neq \emptyset \] Then, there is a tetrahedron \( t \), a 2-simplex \( \sigma \subseteq \partial t \), a component \( K \) of \( S \cap t \), and a component \( \gamma \) of \( \partial D \cap K \) with \( \partial \gamma \subseteq \sigma \). Since \( S \) is 1-normal, the closure \( D' \) of one component of \( K \setminus \gamma \) is a disc with \( \partial D' \subseteq \gamma \cup \sigma \). By \( \gamma \) innermost in \( D \), we can assume that \( D' \cap \partial D = \gamma \). An isotopy of \( (D, \partial D) \) in \( (M \setminus T^1, S \setminus T^1) \) with support in \( U(D') \), moving \( \partial D \) away from \( D' \), reduces \( \#(\partial D \cap T^2) \), in contradiction to our choice. Thus \( \partial D \cap T^2 = \emptyset \).

Now, assume that \( D \cap T^2 \neq \emptyset \). Then, there is a tetrahedron \( t \), a 2-simplex \( \sigma \subseteq \partial t \), and a disc component \( C \) of \( D \cap t \), such that \( C \cap \sigma = \partial C \) is a single circle. There is a ball \( B \subseteq t \) bounded by \( C \) and a disc in \( \sigma \). An isotopy of \( D \) with support in \( U(B) \), moving \( C \) away from \( t \), reduces \( \#(D \cap T^2) \), in contradiction to our choice. Thus \( D \) is contained in a single tetrahedron \( t \). Since \( S \) is 1-normal, \( \partial D \) bounds a disc \( D' \) in \( S \cap t \). An isotopy with support in \( t \) that is constant on \( \partial D \) moves \( D \) to \( D' \), which yields the lemma.

**Corollary 1** Let \( S_0 \subseteq M \) be a 1-normal sphere that splits \( M \), and let \( S \subseteq B^- (S_0) \) be an almost 1-normal sphere disjoint to \( S_0 \) that is split equivalent to \( S_0 \). Then there is a \( T^1 \)-Morse embedding \( J : S^2 \times I \to M \) with \( J(S^2 \times I) = B^+ (S) \cap B^- (S_0) \) and \( c(J, T^1) = 0 \).

**Proof** Let \( X \) be a graph isomorphic to \( S_0 \cap T^2 \). Since \( S \) is a copy of \( S_0 \), there is an embedding \( \varphi : X \times I \to B^+ (S) \cap B^- (S_0) \) with \( \varphi (X^0 \times I) = \varphi (X \times I) \cap T^1 \), \( \varphi (X \times 0) = S_0 \cap T^2 = S_0 \cap \varphi (X \times I) \), and \( \varphi (X \times 1) \) is the union of the normal arcs in \( S \).

Let \( \gamma \subseteq S \cap \varphi (X \times I) \) be a circle that does not meet \( T^1 \). Then, \( \gamma \) bounds a disc \( D \subseteq \varphi (X \times I) \setminus T^1 \). The two components of \( S \setminus \gamma \) are discs. One of them is disjoint to \( T^1 \), since otherwise the disc \( D \) would give rise to a splitting disc for \( S^\times = S_0 \) that is not isotopic mod \( T^1 \) to a sub-disc of \( S_0 \), in contradiction to the preceding lemma. Thus by cut-and-paste along sub-discs of \( S \setminus T^1 \), we can assume that additionally \( S \cap \varphi (X \times I) = \varphi (X \times 1) \).

Let \( \gamma \subseteq X \) be a circle so that \( \varphi (\gamma \times 0) \) is contained in the boundary of a tetrahedron of \( T \). Since \( S_0 \) is 1-normal, \( \varphi (\gamma \times 0) \) bounds an open disc in \( S_0 \setminus T^2 \). By the same argument as in the preceding paragraph, \( \varphi (\gamma \times 1) \) bounds an open disc in \( S \setminus T^1 \). One easily verifies that these two discs together with \( \varphi (\gamma \times I) \) bound a ball in \( B^+ (S) \cap B^- (S_0) \) disjoint to \( T^1 \). Hence \( (B^+ (S) \cap B^- (S_0)) \setminus U(\varphi (X \times I)) \) is a disjoint union of balls in \( M \setminus T^1 \), and this implies the existence of \( J \).

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7 Reduction of surfaces

Let $M$ be a closed connected orientable 3–manifold with a triangulation $T$. In this section, we show how to get isotopies of embedded surfaces under which the number of intersections with $T$ is monotonely non-increasing.

**Definition 15** Let $S \subset M$ be a connected embedded surface that is transversal to $T$ and splits $M$. Let $D$ be an upper (resp. lower) bond of $S$, set $D_1 = U(D) \cap S$, and set $D_2 = B^+(S) \cap \partial U(D)$ (resp. $D_2 = B^-(S) \cap \partial U(D)$). An elementary reduction along $D$ transforms $S$ to the surface $(S \setminus D_1) \cup D_2$.

Upper (resp. lower) reductions of $S$ are the surfaces that are obtained from $S$ by a sequence of elementary reductions along upper (resp. lower) bonds.

If $S'$ is an upper or lower reduction of $S$, then $\|S'\| \leq \|S\|$ with equality if and only if $S = S'$. Obviously $S$ is isotopic to $S'$, such that $\|\cdot\|$ is monotonely non-increasing under the isotopy. If $\alpha \subset T^1 \setminus T^0$ is an arc with $\partial \alpha \subset S'$, then also $\partial \alpha \subset S$. It is easy to see that if $S'$ has a lower compressing disc and is an upper reduction of $S$, then also $S$ has a lower compressing disc.

We will construct surfaces with almost 1–normal upper or lower reductions. Let $N \subset M$ be a 3–dimensional sub–manifold, such that $\partial N$ is pre-normal. Let $S \subset N$ be an embedded surface transversal to $T^2$ that splits $M$ and has no lower compressing disc.

**Lemma 12** Suppose that there is a system $\Gamma \subset N \setminus T^1$ of arcs such that $S^\Gamma \subset N$ is connected, $\Gamma \subset B^-(S^\Gamma)$, and $\partial N \cap B^+(S^\Gamma)$ is 1–normal.

If, moreover, $\Gamma$ and an upper reduction $S' \subset N$ of $S^\Gamma$ are chosen so that $\|S'\|$ is minimal, then $S'$ is almost 1–normal.

**Proof** By hypothesis, $\Gamma \subset B^-(S^\Gamma)$, and $S$ has no lower compressing discs. Thus by Lemma 8, $S^\Gamma$ has no lower compressing discs. Therefore its upper reduction $S'$ has no lower compressing discs.

Assume that $S'$ is not almost 1–normal. Then $S'$ has a compressing disc $D'$ that is contained in a single tetrahedron $t$ (see [16]), with string $\alpha'$ and base $\beta'$. Since $S'$ has no lower compressing discs, $D'$ is upper and does not contain proper compressing sub-discs. Thus $\alpha' \cap S' = \partial \alpha'$, i.e., all components of $(D' \cap S') \setminus \beta'$ are circles. Since $\partial N$ is pre-normal, $\partial N \setminus T^2$ is a disjoint union of discs. Therefore, since $D'$ is contained in a single tetrahedron, we can assume by isotopy of $D'$ mod $T^2$ that $D' \cap \partial N$ consists of arcs. We have

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\[ \alpha' \subset B^+ (S') \subset B^+ (S^\Gamma). \]

It follows that \( \partial N \cap \alpha' = \emptyset \), since otherwise a sub-disc of \( D' \) is a compressing disc for \( \partial N \cap B^+ (S^\Gamma) \), which is impossible as \( \partial N \cap B^+ (S^\Gamma) \) is 1-normal by hypothesis. Thus \( \partial N \cap \alpha' = \emptyset \) and \( D' \subset N \).

By an isotopy with support in \( U(D') \) that is constant on \( \beta' \), we move \( (D' \cap S') \setminus \beta' \to U(D') \setminus t \), and obtain from \( S' \) a surface \((S')^* \subset N\) that has \( D' \) as upper bond. This is shown in Figure 3, where \( B^+ (S') \) is indicated by plus signs and \( T^1 \) is bold. The isotopy moves \( \Gamma \) to a system of arcs \( \Gamma^* N \subset N \) and moves \( S^\Gamma \) to \( S^\Gamma^* \) with \( \Gamma^* \subset B^- (S^\Gamma^*) \). Since \( \alpha' \subset B^+ (S') \), there is a homeomorphism \( \varphi: B^- (S') \to B^- ((S')^*) \) that is constant on \( T^1 \) with \( \varphi(B^- (S^\Gamma)) = B^- (S^\Gamma^*) \).

One obtains \( S' \) by a sequence of elementary reductions along bonds of \( S^\Gamma \) that are contained in \( B^- (S') \). These bonds are carried by \( \varphi \) to bonds of \( S^\Gamma^* \). Thus \( (S')^* \) is an upper reduction of \( S^\Gamma \). Since \((S')^* \) admits an elementary reduction along its upper bond \( D' \), we obtain a contradiction to the minimality of \( \| S' \| \). Thus \( S' \) is almost 1-normal.

**Lemma 13** Let \( \Gamma \) and \( S' \) be as in the previous lemma, and let \( G_1, G_2 \) be two connected components of \( (S')^x \) that both split \( M \). Then there is no arc in \((T^1 \setminus T^0) \cap B^+ (S') \cap N\) joining \( G_1 \) with \( G_2 \).

**Proof** By the previous lemma, \( S' \) is almost 1-normal. Recall that one obtains \( (S')^x \) up to isotopy mod \( T^1 \) by splitting \( S' \) along splitting discs that do not meet \( T^2 \). Assume that there is an arc \( \alpha \subset (T^1 \setminus T^0) \cap B^+ (S') \cap N \) joining \( G_1 \) with \( G_2 \). Let \( Y \) be the component of \( M \setminus (G_1 \cup G_2) \) that contains \( \alpha \).

By hypothesis, \( S^\Gamma \) is connected. Thus \( S' \) is connected, and there is an arc \( \beta \subset S' \) with \( \partial \beta = \partial \alpha \). Since \( G_1, G_2 \) split \( M \), the set \( Y \) is the only component of \( M \setminus (G_1 \cup G_2) \) with boundary \( G_1 \cup G_2 \). Thus there is a component \( \beta' \) of \( \beta \cap Y \) connecting \( G_1 \) with \( G_2 \). There is a splitting disc \( D \subset Y \) of \( S' \) contained in a single tetrahedron with \( \beta' \cap D \neq \emptyset \). By choosing \( D \) innermost, we assume that

\[ \text{Figure 3: How to produce a bond} \]
\( \beta \cap D \) is a single point in \( \partial D \). Since \( \partial N \) is pre-normal and \( D \) is contained in a single tetrahedron, we can assume by isotopy of \( D \) mod \( T^2 \) that \( D \cap \partial N = \emptyset \), thus \( D \subset N \).

Choose a disc \( D' \subset U(\alpha \cup \beta) \cap B^+(S') \) so that \( D' \cap T^1 = \alpha \) and \( D' \cap S' = \beta \setminus U(\partial D) \). Then \( D' \cap \partial N = \emptyset \), since \( U(\alpha \cup \beta) \cap \partial N = \emptyset \). We split \( S' \) along \( D \), pull the two components of \( (S' \cap \partial U(D)) \setminus D \) along \( (\partial D') \setminus (\alpha \cup \beta) \), and reglue. We obtain a surface \( (S')^* \) with \( D' \) as an upper bond.

Since a small collar of \( \partial D \) in \( D \) is in \( B^-(S') \), there is a homeomorphism \( \varphi : B^-(S') \to B^-(((S')^*)) \) that is constant on \( T^1 \). Set \( \Gamma^* = \varphi(\Gamma) \). Then \( \varphi(S^\Gamma) = S^{\Gamma^*} \) with \( \Gamma^* \subset B^-(S^{\Gamma^*}) \). As in the proof of the previous lemma, \( (S')^* \) is an upper reduction of \( S^{\Gamma^*} \), and \( (S')^* \) admits an elementary reduction along \( D' \).

This contradiction to the minimality of \( ||S'|| \) yields the lemma.

8 Proof of Theorem 2

Let \( T \) be a triangulation of \( S^3 \) with a vertex \( x_0 \in T^0 \). Let \( \Sigma \subset S^3 \) be a maximal system of disjoint 1-normal spheres with \( ||\Sigma|| < 2^{185.1(T)^2} \), as given by Construction 1. Construction 2 extends \( \Sigma \) to a system \( \tilde{\Sigma} \subset S^3 \) of disjoint 2-normal spheres that are pairwise non-isotopic mod \( T^2 \), such that

1. any component of \( \tilde{\Sigma} \) has at most one octagon,
2. any component of \( S^3 \setminus \tilde{\Sigma} \) has at most one boundary component that is not 1-normal,
3. if the boundary of a component \( N \) of \( S^3 \setminus \tilde{\Sigma} \) is 1-normal, then \( N \) does not contain 2-normal spheres with exactly one octagon, and
4. \( ||\tilde{\Sigma}|| < 2^{190.1(T)^2} \).

Let \( N \) be a component of \( S^3 \setminus \tilde{\Sigma} \) that is not a regular neighbourhood of a vertex of \( T \). Let \( S_0 \) be the component of \( \partial N \) with \( N \subset B^-(S_0) \), and let \( S_1, \ldots, S_k \) be the other components of \( \partial N \). Since \( \Sigma \) is maximal, any almost 1-normal sphere in \( N \) is a tube sum of copies of \( S_0, S_1, \ldots, S_k \).

Lemma 14 \( N \cap T^0 = \emptyset \).

Proof If \( x \in N \cap T^0 \), then the sphere \( \partial U(x) \subset N \) is 1-normal. It is not isotopic mod \( T^1 \) to a component of \( \partial N \), since \( N \neq U(x) \). This contradicts the maximality of \( \Sigma \).

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Lemma 15 If $\partial N$ is 1-normal, then there is an arc in $T^1 \cap \overline{N}$ that connects two different components of $\partial N \setminus S_0$.

Proof Let $\partial N = S_0 \cup S_1 \cup \cdots \cup S_k$ be 1-normal. We first consider the case where there is an almost 1-normal sphere $S \subseteq S_1 \circ \cdots \circ S_k$ in $\overline{N}$ that has a compressing disc $D$, with string $\alpha$ and base $\beta$. We choose $D$ innermost, so that $\alpha \cap S = \partial \alpha$. In particular, $\alpha \cap \partial N = \partial \alpha$. Assume that $\alpha \not\subset \overline{N}$. Since $\partial D \setminus \alpha \subset \overline{N}$, there is an arc $\beta' \subset D \cap \partial N$ that connects the endpoints of $\alpha$. The sub-disc $D' \subset D$ bounded by $\alpha \cup \beta'$ is a compressing disc for the 1-normal surface $\partial_N$, in contradiction to Lemma 10. By consequence, $\alpha \subset \overline{N}$. Assume that $\partial \alpha$ is contained in a single component of $\partial N \setminus S_0$, say, in $S_1$. By Lemma 10, $D$ is not a compressing disc for $S_1$, hence $\beta \not\subset S_1$. Thus there is a closed line in $S_1 \setminus \beta$ that separates $\partial \alpha$ on $S_1$, but not on $S$. This is impossible as $S$ is a sphere. We conclude that if $S$ has a compressing disc, then there is an arc $\alpha \subset T^1 \cap \overline{N}$ that connects different components of $\partial N \setminus S_0$.

It remains to consider the case where no sphere in $S_1 \circ \cdots \circ S_k$ contained in $\overline{N}$ has a compressing disc. We will show the existence of an almost 2-normal sphere in $N$ with exactly one octagon, using the technique of thin position. This contradicts property (3) of $\Sigma$ (see the begin of this section), and therefore finishes the proof of the lemma. Let $J$: $S^2 \times I \to B^-(S_0)$ be a $T^1$–Morse embedding, such that

1. $J(S^2 \times 0) = S_0$,
2. $J(S^2 \times \frac{1}{2}) \in S_1 \circ \cdots \circ S_k$ (or $\|J(S^2 \times \frac{1}{2})\| = 0$, in the case $\partial N = S_0$),
3. $B^- \left(J(S^2 \times 1)\right) \cap T^1 = \emptyset$, and
4. $\kappa(J)$ is minimal.

Define $S = J(S^2 \times \frac{1}{2})$. Assume that for some $\xi \in I$ there is a pair $D_1, D_2 \subset M$ of nested or independent upper and lower compressing discs for $J_\xi = J(S^2 \times \xi)$. We show that we can assume $D_1, D_2 \subset B^-(S_0)$. Since $S_0$ is 1–normal, it has no compressing discs by Lemma 10. Thus $(D_1 \cup D_2) \cap S_0$ consists of circles. Any such circle bounds a disc in $S_0 \setminus T^1$ by Lemma 11. By cut-and-paste of $D_1 \cup D_2$, we obtain $D_1, D_2 \subset B^-(S_0)$, as claimed. Now, one obtains from $J$ an embedding $J'$: $S^2 \times I \to B^-(S_0)$ with $\kappa(J') < \kappa(J)$ by isotopy along $D_1 \cup D_2$, see [16], [22]. The embedding $J'$ meets conditions (1) and (3) in the definition of $J$. Since $S \subseteq S_1 \circ \cdots \circ S_k$ has no compressing discs by assumption, $S \cap D_i$ consists of circles. Thus $S$ is split equivalent to $J'(S^2 \times \frac{1}{2})$. So $J'$ meets also condition (2), $J'(S^2 \times \frac{1}{2}) \subseteq S_1 \circ \cdots \circ S_k$, in contradiction to the choice of $J$. This disproves the existence of $D_1, D_2$. In conclusion, if $J_\xi$ has upper and lower bonds, then it is impermeable.

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Let $\xi_{\text{max}}$ be the greatest critical parameter of $J$ with respect to $T^1$ in the interval $]0, \frac{1}{2}[$. We have $N \cap T^0 = \emptyset$ by Lemma 14. Hence the critical point corresponding to $\xi_{\text{max}}$ is a point of tangency of $J_{\xi_{\text{max}}}$ to some edge of $T$. By assumption, $S$ has no upper bonds, thus $\|S\| < \|J_{\xi_{\text{max}}-\epsilon}\|$ for sufficiently small $\epsilon > 0$. Let $\xi_{\text{min}} \in I$ be the smallest critical parameter of $J$ with respect to $T^1$. By Lemma 10, $S_0$ has no bonds, thus $\|S_0\| < \|J_{\xi_{\text{min}}+\epsilon}\|$. Therefore there are consecutive critical parameters $\xi_1, \xi_2 \in ]0, \frac{1}{2}[\cup$ such that

$$\|J_{\xi_1-\epsilon}\| < \|J_{\xi_1+\epsilon}\| > \|J_{\xi_2+\epsilon}\|.$$

Thus $J_{\xi_1+\epsilon}$ has both upper and lower bonds, and is therefore impermeable by the preceding paragraph. One component of $J_{\xi_1+\epsilon}$ is a 2-normal sphere in $N$ with exactly one octagon, by Proposition 3. The existence of that 2-normal sphere is a contradiction to the properties of $\Sigma$, which proves the lemma.

We show that some tube sum $S \in S_1 \circ \cdots \circ S_k$ is isotopic to $S_0$ such that $\|\cdot\|$ is monotone under the isotopy. We consider three cases. In the first case, let $\partial N$ be 1-normal.

**Lemma 16** If $\partial N$ is 1-normal, then there is a sphere $S \in S_1 \circ \cdots \circ S_k$ in $N$ with an upper reduction $S' \subset N$ so that there is a $T^1$-Morse embedding $J$: $S^2 \times I \rightarrow S^3$ with $J(S^2 \times I) = B^+(S') \cap B^-(S_0)$ and $c(J, T^1) = 0$.

**Proof** By Lemma 15, there is an arc $\alpha \subset T^1 \cap N$ that connects two components of $\partial N \setminus S_0$, say, $S_1$ with $S_2$. By Lemma 14, $\alpha$ is contained in an edge of $T$. By Lemma 10, the 1-normal surfaces $S_1, \ldots, S_k$ have no lower compressing discs. Let $\Gamma \subset N$ be a system of $k - 1$ arcs, such that the tube sum $S$ of $S_1, \ldots, S_k$ along $\Gamma$ is a sphere and an upper reduction $S' \subset N$ of $S$ minimizes $\|S'\|$. We have $\|S''\| < \|S\|$, since it is possible to choose $\Gamma$ so that $S$ has an upper bond with string $\alpha$. Since $\Gamma \subset B^-(S)$ and by Lemma 12, $S'$ is almost 1-normal.

By the maximality of $\Sigma$, it follows $S' \in n_0S_0 \circ \cdots \circ n_kS_k$ with non-negative integers $n_0, n_1, \ldots, n_k$. Moreover, $n_i \leq 2$ for $i = 0, \ldots, k$ by Lemma 13. Since $S$ separates $S_0$ from $S_1, \ldots, S_k$, so does $S'$. Thus any path connecting $S_0$ with $S_j$ for some $j \in \{1, \ldots, k\}$ intersects $S'$ in an odd number of points. So alternatively $n_0 \in \{0, 2\}$ and $n_i = 1$ for all $i \in \{1, \ldots, k\}$, or $n_0 = 1$ and $n_i \in \{0, 2\}$ for all $i \in \{1, \ldots, k\}$. Since $\|S'\| < \|S^*\|$, it follows $n_0 = 1$ and $n_i = 0$ for $i \in \{1, \ldots, k\}$, thus $(S')^* = S_0$. The existence of a $T^1$-Morse embedding $J$ with the claimed properties follows then by Corollary 1.

The second case is that $S_0$ is 1-normal, and exactly one of $S_1, \ldots, S_k$ contains exactly one octagon, say, $S_1$. The octagon gives rise to an upper bond $D$ of $S_1$. 

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contained in a single tetrahedron. Since \( \partial N \setminus S_1 \) is 1–normal, \( D \subset N \). Thus an elementary reduction of \( S_1 \) along \( D \) transforms \( S_1 \) to a sphere \( F \subset N \). Since \( S_1 \) is impermeable by Proposition 4, \( F \) has no lower compressing disc (such a disc would give rise to a lower compressing disc for \( S_1 \) that is independent from \( D \)).

**Lemma 17** If \( \partial N \setminus S_0 \) is not 1–normal, then there is a sphere \( S \in S_1 \circ \cdots \circ S_k \) in \( N \) with an upper reduction \( S' \subset N \) so that there is a \( T^1 \)–Morse embedding \( J \colon S^2 \times I \to S^3 \) with \( J(S^2 \times \{0\}) \cap B^+(S_0) = \emptyset \) and \( c(J, T^1) = 0 \).

**Proof** We apply Lemma 12 with \( \Gamma = \emptyset \) to \( F, S_2, \ldots, S_k \), and together with the elementary reduction along \( D \) we obtain a sphere \( S \in S_1 \circ S_2 \circ \cdots \circ S_k \) with an almost 1–normal upper reduction \( S' \subset N \). One concludes \( (S')^\times = S_0 \) and the existence of \( J \) as in the proof of the previous lemma.

We come to the third and last case, namely \( S_0 \) has exactly one octagon and \( \partial N \setminus S_0 \) is 1–normal. The octagon gives rise to a lower bond \( D \) of \( S_0 \), that is contained in \( N \) since \( \partial N \setminus S_0 \) is 1–normal. Thus an elementary reduction of \( S_0 \) along \( D \) yields a sphere \( F \subset N \). Since \( S_0 \) is impermeable by Proposition 4, \( F \) has no upper compressing disc, similar to the previous case.

**Lemma 18** If \( S_0 \) is not 1–normal, then there is a lower reduction \( S' \in S_1 \circ \cdots \circ S_k \) of \( S_0 \), with \( S' \subset N \).

**Proof** We apply Lemma 12 with \( \Gamma = \emptyset \) to lower reductions of \( F \), which is possible by symmetry. Thus, together with the elementary reduction along \( D \), there is a lower reduction \( S' \in n_0 S_0 \circ \cdots \circ n_k S_k \) of \( S_0 \), and \( n_0, \ldots, n_k \leq 2 \) by Lemma 13. Since \( S' \subset B^-(F) \) and \( S_0 \subset B^+(F) \), it follows \( n_0 = 0 \). Since \( S' \) separates \( \partial N \cap B^+(F) \) from \( \partial N \cap B^-(F) \), it follows \( n_1, \ldots, n_k \) odd, thus \( n_1 = \cdots = n_k = 1 \).

We are now ready to construct the \( T^1 \)–Morse embedding \( H \colon S^2 \times I \to S^3 \) with \( c(H, T^1) \) bounded in terms of \( t(T) \), thus to finish the proof of Theorems 1 and 2. Let \( x_0 \in T^0 \) be the vertex involved in the definition of \( B^+(\cdot) \). We construct \( H \) inductively as follows.

Choose \( \xi_1 \in ]0, 1[ \) and choose \( H|[0, \xi_1] \) so that \( H_0 \cap T^2 = \emptyset \), \( H_{\xi_1} = \partial U(x_0) \subset \Sigma \), and \( x_0 \) is the only critical point of \( H|[0, \xi_1] \).

For \( i \geq 1 \), let \( H|[0, \xi_i] \) be already constructed. Our induction hypothesis is that \( H_{\xi_i} \in \Sigma \circ S^* \) for some component \( S_0 \) of \( \Sigma \), and moreover for any choice of \( S_0 \) we have \( H_{\xi_i} \subset B^+(S_0) \). Choose \( \xi_{i+1} \in ]\xi_i, 1[ \).

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Assume that $S_0$ is not of the form $S_0 = \partial U(x)$ for a vertex $x \in T^0 \setminus \{x_0\}$. Then, let $N_i$ be the component of $S^3 \setminus \Sigma$ with $N_i \subset B^-(S_0)$ and $\partial N_i = S_0 \cup S_1 \cup \cdots \cup S_k$ for $S_1, \ldots, S_k \subset \Sigma$. If $S_0$ is 1–normal, then let $S = S_1 \circ \cdots \circ S_k$, $S'$ and $J$ be as in Lemmas 16 and 17. Then, we extend $H[0, \xi_i]$ to $H[0, \xi_{i+1}]$ induced by the embedding $J$, relating $S_0$ with $S'$, and by the inverses of the elementary upper reductions, relating $S'$ with $S$. If $S_0$ is not 1–normal, then let $S = S_1 \circ \cdots \circ S_k$ be as in Lemma 18. We extend $H[0, \xi_i]$ to $H[0, \xi_{i+1}]$ along the elementary lower reductions, relating $S_0$ with $S$. In either case, $H_{\xi_{i+1}} \in S_1 \circ \cdots \circ S_k \circ S^\ast$. The critical points of $H[\xi_i, \xi_{i+1}]$ are contained in $N_i$, given by elementary reductions. Thus the number of these critical points is $\leq \frac{1}{2} \min \{\|S_0\|, \|S\\} \leq \frac{1}{2} \|\Sigma\| < 2^{190 t(T)^2}$, by Construction 2. Since $H_{\xi_{i+1}} \subset B^+(S_m)$ for any $m = 1, \ldots, k$, we can proceed with our induction.

After at most $\#(\Sigma)$ steps, we have $H_{\xi_i} = \partial U(T^0 \setminus \{x_0\})$. Then, choose $H[\xi, 1]$ so that $H \cap T^2 = \emptyset$ and the set of its critical points is $T^0 \setminus \{x_0\}$. By Proposition 2 holds $\#(\Sigma) \leq 10 t(T)$. Thus finally

$$c(H, T^1) < \#(T^0) + 10 t(T) \cdot 2^{190 t(T)^2} < 2^{196 t(T)^2}.$$

9 Proof of Propositions 3 and 4

Let $M$ be a closed connected 3–manifold with a triangulation $T$. We prove Proposition 3, that states that any impermeable surface in $M$ is isotopic mod $T^1$ to an almost 2–normal surface with exactly one octagon. The proof consists of the following three lemmas.

Lemma 19 Any impermeable surface in $M$ is almost 2–normal, up to isotopy mod $T^1$.

Proof We give here just an outline. A complete proof can be found in [16]. Let $S \subset M$ be an impermeable surface. By definition, it has upper and lower bonds with strings $\alpha_1, \alpha_2$. By isotopies mod $T^1$, one obtains from $S$ two surfaces $S_1, S_2 \subset M$, such that $S_i$ has a return $\beta_i \subset T^2$ with $\partial \beta_i = \partial \alpha_i$, for $i \in \{1, 2\}$. A surface that has both upper and lower returns admits an independent pair of upper and lower compressing disc, thus is not impermeable. By consequence, under the isotopy mod $T^1$ that relates $S_1$ and $S_2$ occurs a surface $S'$ that has no returns at all, thus is almost $k$–normal for some natural number $k$.

If there is a boundary component $\zeta$ of a component of $S' \setminus T^2$ and an edge $e$ of $T$ with $\#(\zeta \cap e) > 2$, then there is an independent pair of upper and lower compressing discs. Thus $k = 2$. 

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**Lemma 20** Let $S \subset M$ be an almost 2-normal impermeable surface. Then $S$ contains at most one octagon.

**Proof** Two octagons in different tetrahedra of $T$ give rise to a pair of independent upper and lower compressing discs for $S$. Two octagons in one tetrahedron of $T$ give rise to a pair of nested upper and lower compressing discs for $S$. Both is a contradiction to the impermeability of $S$. \(\square\)

**Lemma 21** Let $S \subset M$ be an almost 2-normal impermeable surface. Then $S$ contains at least one octagon.

**Proof** By hypothesis, $S$ has both upper and lower bonds. Assume that $S$ does not contain octagons, i.e., it is almost 1-normal. We will obtain a contradiction to the impermeability of $S$ by showing that $S$ has a pair of independent or nested compressing discs.

According to Lemma 10, we can assume that $S$ has a compressing disc $D_1$ with string $\alpha_1$ that is contained in a single closed tetrahedron $t_1$. Choose $D_1$ innermost, i.e., $\alpha_1 \cap S = \partial \alpha_1$. Without assumption, let $D_1$ be upper. Since $S$ has no octagon by assumption, $\alpha_1$ connects two different components $\zeta_1, \eta_1$ of $S \setminus \partial t_1$. Let $D$ be a lower bond of $S$. Choose $S, D_1$ and $D$ so that, in addition, $\#(D \cap T^2)$ is minimal.

Let $C$ be the closure of an innermost component of $D \setminus T^2$, which is a disc. There is a closed tetrahedron $t_2$ of $T$ and a closed 2-simplex $\sigma_2 \subset \partial t_2$ of $T$ such that $\partial C \cap \partial t_2$ is a single component $\gamma \subset \sigma_2$. We have to consider three cases.

1. Let $\gamma$ be a circle, thus $\partial C = \gamma$. There is a disc $D' \subset \sigma_2$ with $\partial D' = \gamma$ and a ball $B \subset t_2$ with $\partial B = C \cup D'$. We move $S \cup D$ away from $B$ by an isotopy mod $T^1$ with support in $U(B)$, and obtain a surface $S^*$ with a lower bond $D^*$. As $D$ is a bond, $S \cap D'$ consists of circles. Therefore the normal arcs of $S \cap T^2$ are not changed under the isotopy, and the isotopy does not introduce returns, thus $S^*$ is almost 1-normal. Since $\xi_1 \cap D' = \eta_1 \cap D' = \emptyset$ and $C \cap S = \emptyset$, it follows $B \cap \partial D_1 = \emptyset$. Thus $D_1$ is an upper compressing disc for $S^*$, and $\#(D^* \cap T^2) < \#(D \cap T^2)$ in contradiction to our choice.

2. Let $\gamma$ be an arc with endpoints in a single component $c$ of $S \cap \sigma$. By an isotopy mod $T^1$ with support in $U(C)$ that moves $C$ into $U(C) \setminus t_2$, we obtain from $S$ and $D$ a surface $S^*$ with a lower bond $D^*$. Since $D$ is a bond, the isotopy does not introduce returns, thus $S^*$ is almost 1-normal.
One component of $S^* \cap t_1$ is isotopic mod $T^2$ to the component of $S \cap t_1$ that contains $\partial D_1 \cap S$. Thus up to isotopy mod $T^2$, $D_1$ is an upper compressing disc for $S^*$, and $\#(D^* \cap T^2) < \#(D \cap T^2)$ in contradiction to our choice.

(3) Let $\gamma$ be an arc with endpoints in two different components $c_1, c_2$ of $S \cap \sigma$. Assume that, say, $c_1$ is a circle. By an isotopy mod $T^1$ with support in $U(C)$ that moves $C$ into $U(C) \setminus t_2$, we obtain from $S$ and $D$ a surface $S^*$ with a lower bond $D^*$. Since $D$ is a bond, the isotopy does not introduce returns, thus $S^*$ is almost 1–normal. There is a disc $D' \subset \sigma$ with $\partial D' = c_1$. Let $K$ be the component of $S \cap t_1$ that contains $\partial D_1 \cap S$. One component of $S^* \cap t_1$ is isotopic mod $T^2$ either to $K$ or, if $\partial D' \cap \partial K \neq \emptyset$, to $K \cup D'$. In either case, $D_1$ is an upper compressing disc for $S^*$, up to isotopy mod $T^2$. But $\#(D^* \cap T^2) < \#(D \cap T^2)$ in contradiction to our choice. Thus, $c_1$ and $c_2$ are normal arcs.

Since $S$ is almost 1–normal, $c_1, c_2$ are contained in different components $\zeta_2, \eta_2$ of $S \cap \partial t_2$. By $D$ is a lower bond, $\partial(C \cap D_1) \subset \partial C \cap S$. There is a sub-arc $\alpha_2$ of an edge of $t_2$ and a disc $D' \subset \sigma$ with $\partial D' \subset \alpha_2 \cup \gamma \cup \zeta_2 \cup \eta_2$ and $\alpha_2 \cap S = \partial \alpha_2$. The disc $D_2 = C \cup D' \subset t_2$ is a lower compressing disc for $S$ with string $\alpha_2$, and $\partial(D_1 \cap D_2) \subset \partial D_2 \cap S$. At least one component of $\partial t_1 \setminus (\zeta_1 \cup \eta_1)$ is a disc that is disjoint to $D_2$. Let $D_m$ be the closure of a copy of such a disc in the interior of $t_1$, with $\partial D_m \subset S$. By construction, $D_1 \cap D_m = \partial D_1 \cap \partial D_m$ is a single point and $D_2 \cap D_m = \emptyset$. Thus by Lemma 9, $S$ has a pair of independent or nested upper and lower compressing discs and is therefore not impermeable. \qed

Proof of Proposition 4 Let $S \subset M$ be a connected 2–normal surface that splits $M$, and assume that exactly one component $O$ of $S \setminus T^2$ is an octagon. The octagon gives rise to upper and lower bonds of $S$.

Let $D_1, D_2$ be any upper and lower compressing discs for $S$. We have to show that $D_1$ and $D_2$ are neither impermeable nor nested. It suffices to show that $\partial D_1 \cap \partial D_2 \not\subset T^1$. To obtain a contradiction, assume that $\partial D_1 \cap \partial D_2 \subset T^1$. Choose $D_1, D_2$ so that $\#(\partial D_1 \setminus T^2) + \#(\partial D_2 \setminus T^2)$ is minimal.

Let $t$ be a tetrahedron of $T$ with a closed 2–simplex $\sigma \subset \partial t$, and let $\beta$ be a component of $\partial D_1 \cap t$ (resp. $\partial D_2 \cap t$) such that $\partial \beta$ is contained in a single component of $S \cap \sigma$. Since $S$ is 2–normal, there is a disc $D \subset S \cap t$ and an arc $\gamma \subset S \cap \sigma$ with $\partial D = \beta \cup \gamma$. By choosing $\beta$ innermost in $D$, we can assume that $D \cap (\partial D_1 \cup \partial D_2) = \beta$. An isotopy of $(D_1, \partial D_1)$ (resp. $(D_2, \partial D_2)$) in $(M, S)$ with support in $U(D)$ that moves $\beta$ to $U(D) \setminus t$ reduces $\#(\partial D_1 \setminus T^2)$ (resp.
The size of triangulations supporting a given link

#(\partial D_2 \setminus T^2)), leaving \partial D_1 \cap \partial D_2 unchanged. This is a contradiction to the minimality of \(D_1, D_2\).

For \(i = 1, 2\), there are arcs \(\beta_i \subset \partial D_i \setminus T^1\) and \(\gamma_i \subset D_i \cap T^2\) such that \(\beta_i \cup \gamma_i\) bounds a component of \(D_i \setminus T^2\), by an innermost arc argument. Let \(t_i\) be the tetrahedron of \(T\) that contains \(\beta_i\), and let \(\sigma_i \subset \partial t_i\) be the close 2-simplex that contains \(\gamma_i\). We have seen above that \(\partial \beta_i\) is not contained in a single component of \(S \cap \sigma_i\). Since \(S\) is 2-normal, i.e., has no tubes, it follows that \(\beta_i \subset O\). Since collars of \(\beta_1\) in \(D_1\) and of \(\beta_2\) in \(D_2\) are in different components of \(t \setminus O\), it follows \(\beta_1 \cap \beta_2 \neq \emptyset\). Thus \(\partial D_1 \cap \partial D_2 \notin T^1\), which yields Proposition 4.

References


