We study minimal solutions for one-dimensional variational problems on a torus. We show that, for a generic integrand and any rational number $\alpha$, there exists a unique (up to translations) periodic minimal solution with rotation number $\alpha$.

1. Introduction

In this paper, we consider functionals of the form

$$I^f(a,b,x) = \int_a^b f(t,x(t),x'(t)) \, dt,$$

where $a$ and $b$ are arbitrary real numbers satisfying $a < b$, $x \in W^{1,1}(a,b)$ and $f$ belongs to a space of functions described below. By an appropriate choice of representatives, $W^{1,1}(a,b)$ can be identified with the set of absolutely continuous functions $x : [a,b] \to \mathbb{R}$, and henceforth we will assume that this has been done.

Denote by $\mathcal{M}$ the set of integrands $f = f(t,x,p) : \mathbb{R}^3 \to \mathbb{R}$ which satisfy the following assumptions:

(A1) $f \in C^3$ and $f(t,x,p)$ has period 1 in $t,x$;

(A2) $\delta_f \leq f_{pp}(t,x,p) \leq \delta_f^{-1}$ for every $(t,x,p) \in \mathbb{R}^3$;

(A3) $|f_{xp}| + |f_{tp}| \leq c_f(1 + |p|), |f_{xx}p| + |f_{xp}| \leq c_f(1 + p^2),$

with some constants $\delta_f \in (0,1)$, $c_f > 0$.

Clearly, these assumptions imply that

$$\delta_f p^2 - \tilde{c}_f \leq f(t,x,p) \leq \delta_f^{-1} p^2 + \tilde{c}_f$$

for every $(t,x,p) \in \mathbb{R}^3$ for some constants $\tilde{c}_f > 0$ and $0 < \tilde{\delta}_f < \delta_f$.

In this paper, we analyse extremals of variational problems with integrands $f \in \mathcal{M}$. The following optimality criterion was introduced by Aubry and Le
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Let $f \in \mathcal{M}$. A function $x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ is called an $(f)$-minimal solution if

$$I^f(a, b, y) \geq I^f(a, b, x)$$

for each pair of numbers $a < b$ and each $y \in W^{1,1}(a, b)$ which satisfies $y(a) = x(a)$ and $y(b) = x(b)$ (see [2, 9, 10, 12]).

Our work follows Moser [9, 10], who studied the existence and structure of minimal solutions in the spirit of Aubry-Mather theory [2, 7].

Consider any $f \in \mathcal{M}$. It was shown in [9, 10] that $(f)$-minimal solutions possess numerous remarkable properties. Thus, for every $(f)$-minimal solution $x(\cdot)$, there is a real number $\alpha$ satisfying

$$\sup \{|x(t) - at| : t \in \mathbb{R}^1\} < \infty$$

which is called the rotation number of $x(\cdot)$, and given any real $\alpha$ there exists an $(f)$-minimal solution with rotation number $\alpha$. Senn [11] established the existence of a strictly convex function $E_f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$, which is called the minimal average action of $f$ such that, for each real $\alpha$ and each $(f)$-minimal solution $x(\cdot)$ with rotation number $\alpha$,

$$(T_2 - T_1)^{-1} I^f(T_1, T_2, x) \longrightarrow E_f(\alpha) \quad \text{as} \quad T_2 - T_1 \longrightarrow \infty.$$ (1.5)

This result is an analogue of Mather’s theorem about the average energy function for Aubry-Mather sets generated by a diffeomorphism of the infinite cylinder [8].

In this paper, we show that for a generic integrand $f$ and any rational $\alpha$, there exists a unique (up to translations) $(f)$-minimal periodic solution with rotation number $\alpha$.

Let $k \geq 3$ be an integer. Set $\mathcal{M}_k = \mathcal{M} \cap C^k(\mathbb{R}^3)$. For $f \in \mathcal{M}_k$ and $q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3$ satisfying $q_1 + q_2 + q_3 \leq k$, we set

$$|q| = q_1 + q_2 + q_3, \quad D^q f = \frac{\partial |q| f}{\partial t^{q_1} \partial x^{q_2} \partial p^{q_3}}.$$ (1.6)

For $N, \varepsilon > 0$ we set

$$E_k(N, \varepsilon) = \{ (f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)|$$

$$\leq \varepsilon + \varepsilon \max \{|D^q f(t, x, p)|, |D^q g(t, x, p)|\}$$

$$\forall q \in \{0, 1, 2\}^3 \text{ satisfying } |q| \in \{0, 2\}, \forall (t, x, p) \in \mathbb{R}^3 \}$$

$$\cap \{ (f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \leq \varepsilon$$

$$\forall q \in \{0, \ldots, k\}^3 \text{ satisfying } |q| \leq k, \forall (t, x, p) \in \mathbb{R}^3$$

such that $|p| \leq N \}.$$ (1.7)
It is easy to verify that, for the set \( \mathcal{M}_k \) there exists a uniformity which is determined by the base \( E_k(N,\varepsilon) \), \( N,\varepsilon > 0 \), and that the uniform space \( \mathcal{M}_k \) is metrizable and complete [3]. We establish the existence of a set \( \mathcal{F}_k \subset \mathcal{M}_k \) which is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \) such that, for each \( f \in \mathcal{F}_k \) and each rational \( \alpha \in \mathbb{R}^1 \), there exists a unique (up to translations) \((f)\)-minimal periodic solution with rotation number \( \alpha \).

2. Properties of minimal solutions

Consider any \( f \in \mathcal{M} \). We note that, for each pair of integers \( j \) and \( k \) the translations \( (t,x) \rightarrow (t+j,x+k) \) leave the variational problem invariant. Therefore, if \( x(\cdot) \) is an \((f)\)-minimal solution, so is \( x(\cdot+j)+k \). Of course, on the torus, this represents the same curve as does \( x(\cdot) \). This motivates the following terminology [9, 10].

We say that a function \( x(\cdot) \in W^{1,1}_{loc}(\mathbb{R}^1) \) has no self-intersections if for all pairs of integers \( j,k \) the function \( t \rightarrow x(t+j)+k-x(t) \) is either always positive, or always negative, or identically zero.

Denote by \( \mathbb{Z} \) the set of all integers. We have the following result (see [6, Proposition 3.2] and [9, 10]).

**Proposition 2.1.** (i) Let \( f \in \mathcal{M} \). Given any real \( \alpha \) there exists a nonself-intersecting \((f)\)-minimal solution with rotation number \( \alpha \).

(ii) For any \( f \in \mathcal{M} \) and any \((f)\)-minimal solution \( x \), there is the rotation number of \( x \).

For each \( f \in \mathcal{M} \), each rational number \( \alpha \), and each natural number \( q \) satisfying \( qa \in \mathbb{Z} \), we define

\[
N(\alpha, q) = \{ x(\cdot) \in W^{1,1}_{loc}(\mathbb{R}^1) : x(t+q) = x(t) + qa, \ t \in \mathbb{R}^1 \},
\]

\[
M_f(\alpha, q) = \{ x(\cdot) \in N(\alpha, q) : I_f(0,q,x) \leq I_f(0,q,y) \ \forall y \in N(\alpha, q) \}. \tag{2.1}
\]

We have the following result [9, Theorems 5.1, 5.2, 5.4, and Corollaries 5.3 and 5.5].

**Proposition 2.2.** Let \( f \in \mathcal{M} \), let \( \alpha \) be a rational number, and let \( p,q \geq 1 \) be integers satisfying \( pq, qa \in \mathbb{Z} \). Then \( M_f(\alpha, q) = M_f(\alpha, p) \neq \emptyset \), each \( x \in M_f(\alpha, q) \) is a nonself-intersecting \((f)\)-minimal solution with rotation number \( \alpha \) and the set \( M_f(\alpha, q) \) is totally ordered, that is, if \( x, y \in M_f(\alpha, q) \), then either \( x(t) < y(t) \) for all \( t \), or \( x(t) > y(t) \) for all \( t \), or \( x(t) = y(t) \) identically.

For any \( f \in \mathcal{M} \) and any rational number \( \alpha \) we set \( M^\text{per}_f(\alpha) = M_f(\alpha, q) \), where \( q \) is a natural number satisfying \( qa \in \mathbb{Z} \).

We have the following result (see [6, Theorem 1.1]).

**Proposition 2.3.** Let \( f \in \mathcal{M} \). Then there exist a strictly convex function \( E_f : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) satisfying \( E_f(\alpha) \rightarrow \infty \) as \( |\alpha| \rightarrow \infty \) and a monotonically increasing function \( \Gamma_f : (0, \infty) \rightarrow [0, \infty) \) such that for each real \( \alpha \), each \((f)\)-minimal solution \( x \) with
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rotation number \( \alpha \) and each pair of real numbers \( S \) and \( T \),

\[
|I^I (S, S + T, x) - E_f (\alpha) T| \leq \Gamma_f (|\alpha|).
\] (2.2)

By Proposition 2.3 for each \( f \in \mathcal{M} \) there exists a unique number \( \alpha (f) \) such that

\[
E_f (\alpha (f)) = \min \{ E_f (\beta) : \beta \in \mathbb{R}^1 \}.
\] (2.3)

Note that assumptions (A1), (A2), and (A3) play an important role in the proofs of Propositions 2.1, 2.2, and 2.3 (see [9, 10]).

3. The main results

**Theorem 3.1.** Let \( k \geq 3 \) be an integer and \( \alpha \) be a rational number. Then there exists a set \( \mathcal{F} \subset \mathcal{M}_k \) which is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \) such that for each \( f \in \mathcal{M}_k \) the following assertions hold:

1. If \( x, y \in \mathcal{M}_f (\alpha) \), then there are integers \( p, q \) such that \( y(t) = x(t + p) - q \) for all \( t \in \mathbb{R}^1 \).
2. Let \( x \in \mathcal{M}_f (\alpha) \) and \( \epsilon > 0 \). Then there exists a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{M}_k \) such that for each \( g \in \mathcal{U} \) and each \( y \in \mathcal{M}_g (\alpha) \) there are integers \( p, q \) such that \( |y(t) - x(t + p) + q| \leq \epsilon \) for all \( t \in \mathbb{R}^1 \).

It is not difficult to see that Theorem 3.1 implies the following result.

**Theorem 3.2.** Let \( k \geq 3 \) be an integer. Then there exists a set \( \mathcal{F} \subset \mathcal{M}_k \) which is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \) such that, for each \( f \in \mathcal{M}_k \) and each rational number \( \alpha \) the assertions (1) and (2) of Theorem 3.1 hold.

Note that minimal solutions with irrational rotation numbers were studied in [2, 7, 9, 10, 12].

4. An auxiliary result

Let \( k \geq 3 \) be an integer and \( \beta \in \mathbb{R}^1 \). For each \( f \in \mathcal{M}_k \), define \( \mathcal{A} f \in C^3 (\mathbb{R}^3) \) by

\[
(\mathcal{A} f) (t, x, u) = f (t, x, u) - \beta u, \quad (t, x, u) \in \mathbb{R}^3.
\] (4.1)

Clearly \( \mathcal{A} f \in \mathcal{M}_k \) for each \( f \in \mathcal{M}_k \).

**Proposition 4.1.** The mapping \( \mathcal{A} : \mathcal{M}_k \rightarrow \mathcal{M}_k \) is continuous.

**Proof.** Let \( f \in \mathcal{M}_k \) and let \( N, \epsilon > 0 \). In order to prove the proposition, it is sufficient to show that there exists \( \epsilon_0 \in (0, \epsilon) \) such that

\[
\mathcal{A} \left( \{ g \in \mathcal{M}_k : (f, g) \in E_k (N, \epsilon_0) \} \right) \subset \{ h \in \mathcal{M}_k : (h, \mathcal{A} f) \in E_k (N, \epsilon) \}. \tag{4.2}
\]

Set

\[
\Delta_0 = 2(|\beta| + 1). \tag{4.3}
\]
Equation (1.2) implies that there exists $c_0 > 0$ such that

$$\Delta_0 |u| - c_0 \leq f(t, x, u) \quad \forall (t, x, u) \in \mathbb{R}^3. \quad (4.4)$$

Choose a number $\varepsilon_0$ such that

$$0 < \varepsilon_0 < \min\{1, \epsilon\}, \quad 4\varepsilon_0 + 4\varepsilon_0(1 - \varepsilon_0)^{-1}(4 + c_0) < \epsilon. \quad (4.5)$$

It follows from (4.3) and (4.4) that for each $(t, x, u) \in \mathbb{R}^3$,

$$|f(t, x, u) - \beta u| \geq |f(t, x, u)| - |\beta u| \geq |f(t, x, u)| - |\beta|\Delta_0^{-1}(f(t, x, u) + c_0)$$

$$\geq |f(t, x, u)|(1 - |\beta|\Delta_0^{-1}) - |\beta|\Delta_0^{-1}c_0$$

$$\geq 2^{-1}|f(t, x, u)| - 2^{-1}c_0. \quad (4.6)$$

Assume that

$$g \in \mathcal{M}_k, \quad (f, g) \in E_k(N, \varepsilon_0). \quad (4.7)$$

By (1.7) and (4.7) for each $(t, x, u) \in \mathbb{R}^3$,

$$|f(t, x, u) - g(t, x, u)| \leq \varepsilon_0 + \varepsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\},$$

$$\max\{|f(t, x, u)|, |g(t, x, u)|\} - \min\{|f(t, x, u)|, |g(t, x, u)|\} \leq \varepsilon_0 + \varepsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\},$$

$$(1 - \varepsilon_0) \max\{|f(t, x, u)|, |g(t, x, u)|\} \leq \min\{|f(t, x, u)|, |g(t, x, u)|\} + \varepsilon_0,$$

$$|g(t, x, u)| \leq (1 - \varepsilon_0)^{-1}|f(t, x, u)| + (1 - \varepsilon_0)^{-1}\varepsilon_0. \quad (4.8)$$

We show that $(\mathcal{A}f, \mathcal{A}g) \in E_k(N, \epsilon)$. It follows from (1.7), (4.1), (4.5), and (4.7) that, for each $q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3$ satisfying $|q| \leq k$ and each $(t, x, p) \in \mathbb{R}^3$ satisfying $|p| \leq N$,

$$|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)| \leq \varepsilon_0 < \epsilon. \quad (4.9)$$

Let $q \in \{0, 1, 2\}^3$, $|q| \in \{0, 2\}$, and $(t, x, p) \in \mathbb{R}^3$. Equation (4.1) implies that

$$|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)|. \quad (4.10)$$

If $|q| = 2$, then by (1.7), (4.1), (4.5), (4.7), and (4.10),

$$|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| \leq \varepsilon_0 + \varepsilon_0 \max\{|D^q f(t, x, p)|, |D^q g(t, x, p)|\}$$

$$< \epsilon + \epsilon \max\{|D^q(\mathcal{A}f)(t, x, p)|, |D^q(\mathcal{A}g)(t, x, p)|\}. \quad (4.11)$$
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Assume that $q = 0$. By (1.7), (4.1), (4.5), (4.6), (4.7), and (4.8),

\[
|D^q(\mathcal{A}f)(t,x,p) - D^q(\mathcal{A}g)(t,x,p)| = |f(t,x,p) - g(t,x,p)| \\
\leq \varepsilon_0 + \varepsilon_0 \max \{ |f(t,x,p)|, |g(t,x,p)| \} \\
\leq \varepsilon_0 + \varepsilon_0 \max \{ |f(t,x,p)|, (1 - \varepsilon_0)^{-1} |f(t,x,p)| + (1 - \varepsilon_0)^{-1} \varepsilon_0 \} \\
= \varepsilon_0 + \varepsilon_0 (1 - \varepsilon_0)^{-1} |f(t,x,p)| + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} \\
\leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + \varepsilon_0 (1 - \varepsilon_0)^{-1} [2 |f(t,x,p) - \beta p| + c_0] \\
\leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} c_0 + 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} |f(t,x,p) - \beta p| \\
\leq 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} |(\mathcal{A}f)(t,x,p)| + \varepsilon \leq \varepsilon + \varepsilon |(\mathcal{A}f)(t,x,p)|.
\]

Equations (4.9), (4.11), and (4.12) imply that $(\mathcal{A}f, \mathcal{A}g) \in E_k(N, \varepsilon)$. Proposition 4.1 is proved. □

Let $-\infty < T_1 < T_2 < \infty$ and $x \in W^{1,1}(T_1, T_2)$. By (4.1) we have

\[
I^{\mathcal{A}f}(T_1, T_2, x) = \int_{T_1}^{T_2} (f(t, x(t), x'(t)) - \beta x'(t)) dt \\
= I^f(T_1, T_2, x) - \beta x(T_2) + \beta x(T_1).
\]

Therefore, each $x \in W^{1,1}_{loc}(\mathbb{R}^1)$ is an $(\mathcal{A}f)$-minimal solution if and only if $x(\cdot)$ is an $(f)$-minimal solution.

Let $x \in W^{1,1}_{loc}(\mathbb{R}^1)$ be an $(f)$-minimal solution with rotation number $r$. By Proposition 2.1 there exists $c_1 > 0$ such that for all $s, t \in \mathbb{R}^1$,

\[
|x(t+s) - x(t) - rs| \leq c_1. \tag{4.14}
\]

Proposition 2.3 implies that there exists a constant $c_2 > 0$ such that for each $s \in \mathbb{R}^1$ and each $t > 0$,

\[
|I^f(s, s+t, x) - E_f(r)t| \leq c_2, \tag{4.15}
\]

\[
|I^{\mathcal{A}f}(s, s+t, x) - E_{\mathcal{A}f}(r)t| \leq c_2. \tag{4.16}
\]

It follows from (4.13), (4.14), (4.15), and (4.16) that, for each $s \in \mathbb{R}^1$ and each $t > 0$,

\[
|E_{\mathcal{A}f}(r)t + \beta tr - E_f(r)t| \\
\leq |E_{\mathcal{A}f}(r)t - I^{\mathcal{A}f}(s, s+t, x)| + |I^{\mathcal{A}f}(s, s+t, x) + \beta tr - I^f(s, s+t, x)| \\
+ |I^f(s, s+t, x) - E_f(r)t| \\
\leq c_2 + |\beta tr - \beta [x(t+s) - x(s)]| + \varepsilon c_2 \leq 2c_2 + |\beta| c_1.
\]

These inequalities imply that

\[
E_{\mathcal{A}f}(r) = E_f(r) - \beta r \quad \forall r \in \mathbb{R}^1. \tag{4.18}
\]
5. Proof of Theorem 3.1

Let \( g \in \mathcal{M} \). We define

\[
\mu(g) = \inf \left\{ \liminf_{T \to \infty} T^{-1} I^g(0, T, x) : x(\cdot) \in W^{1,1}_{loc}([0, \infty)) \right\}. \tag{5.1}
\]

In [13, Section 5] we showed that the number \( \mu(g) \) is well defined and proved the following result [13, Theorem 5.1].

**Proposition 5.1.** Let \( f \in \mathcal{M} \). Then there exists a constant \( M_0 > 0 \) such that:

(i) \( I^f(0, T, x) - \mu(f) T \geq -M_0 \) for each \( x \in W^{1,1}_{loc}([0, \infty)) \) and each \( T > 0 \).

(ii) For each \( a \in \mathbb{R}^1 \) there exists \( x \in W^{1,1}_{loc}([0, \infty)) \) such that \( x(0) = a \) and

\[
|I^f(0, T, x) - \mu(f) T| \leq 4M_0 \quad \forall T > 0. \tag{5.2}
\]

Note that assertion (ii) of Proposition 5.1 holds by the periodicity of \( f \) in \( x \).

Let \( f \in \mathcal{M} \). A function \( x \in W^{1,1}_{loc}([0, \infty)) \) is called \((f)\)-good (see [5]) if

\[
\sup \{ |I^f(0, T, x) - \mu(f) T| : T \in (0, \infty) \} < \infty. \tag{5.3}
\]

By [6, Theorem 4.1],

\[
E_f(\alpha(f)) = \mu(f) \quad \forall f \in \mathcal{M}. \tag{5.4}
\]

For \( f \in \mathcal{M} \), \( x, y, T_1 \in \mathbb{R}^1 \), and \( T_2 > T_1 \) we set

\[
U^f(T_1, T_2, x, y) = \inf \{ I^f(T_1, T_2, v) : v \in W^{1,1}(T_1, T_2), v(T_1) = x, v(T_2) = y \}. \tag{5.5}
\]

It is not difficult to see that for each \( x, y, T_1 \in \mathbb{R}^1 \), \( T_2 > T_1 \),

\[
U^f(T_1, T_2, x+1, y+1) = U^f(T_1, T_2, x, y),
\]

\[
U^f(T_1 + 1, T_2 + 1, x, y) = U^f(T_1, T_2, x, y), \quad -\infty < U^f(T_1, T_2, x, y) < \infty,
\]

\[
\inf \{ U^f(T_1, T_2, a, b) : a, b \in \mathbb{R}^1 \} > -\infty. \tag{5.6}
\]

Denote by \( \mathcal{M}_{per} \) the set of all \( f \in \mathcal{M} \) such that \( \alpha(f) \) is rational and denote by \( \mathcal{M}_{per}^0 \) the set of all \( g \in \mathcal{M}_{per} \) for which there exist an \((g)\)-minimal solution \( w \in C^2(\mathbb{R}^1) \), a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \), and integers \( m, n \) such that the following properties hold:

(P1) \( \pi(x+1) = \pi(x), x \in \mathbb{R}^1; \)

(P2) \( n \geq 1 \) and \( \alpha(g) = mn^{-1} \) is an irreducible fraction;

(P3) \( w(t+n) = w(t) + m \) for all \( t \in \mathbb{R}^1; \)

(P4) \( U^g(0, 1, x, y) - \mu(g) - \pi(x) + \pi(y) \geq 0 \) for each \( x, y \in \mathbb{R}^1; \)

(P5) for any \( u \in W^{1,1}(0, n) \), the equality

\[
I^g(0, n, u) = n\mu(g) + \pi(u(0)) - \pi(u(n)) \tag{5.7}
\]

holds if and only if there are integers \( i, j \) such that \( u(t) = w(t+i) - j \) for all \( t \in [0, n] \).
Consider the manifold \( (\mathbb{R}^1/\mathbb{Z})^2 \) and the canonical mapping \( P : \mathbb{R}^2 \to (\mathbb{R}^1/\mathbb{Z})^2 \). We have the following result [13, Proposition 6.2].

**Proposition 5.2.** Let \( \Omega \) be a closed subset of \( (\mathbb{R}^1/\mathbb{Z})^2 \). Then there exists a bounded nonnegative function \( \phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2) \) such that

\[
\Omega = \{ x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0 \}.
\]

(5.8)

Proposition 5.2 is proved by using [1, Chapter 2, Section 3, Theorem 1] and the partition of unity (see [4, Appendix 1]).

We also have the following result (see [13, Proposition 6.3]).

**Proposition 5.3.** Suppose that \( f \in \mathcal{M}_{\text{per}} \), \( \alpha(f) = mn^{-1} \) is an irreducible fraction (\( m, n \) are integers, \( n \geq 1 \)) and \( w \in W^{1,1}_\text{loc}(\mathbb{R}) \) is an \( f \)-minimal solution satisfying \( w(t + n) = w(t) + m \) for all \( t \in \mathbb{R} \). Let \( \phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2) \) be as guaranteed in Proposition 5.2 with

\[
\Omega = \{ P(t, w(t)) : t \in [0, n] \},
\]

(5.9)

and let

\[
g(t, x, p) = f(t, x, p) + \phi(P(t, x)), \quad (t, x, p) \in \mathbb{R}^3.
\]

(5.10)

Then \( g \in \mathcal{M}^0_{\text{per}} \) and there is a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \) such that the properties (P1), (P2), (P3), (P4), and (P5) hold with \( g, w, \pi, m, n \) and \( \alpha(g) = \alpha(f) \).

In the sequel we need the following two lemmas proved in [13].

**Lemma 5.4** [13, Lemma 6.6]. Assume that \( k \geq 3 \) is an integer, \( g \in \mathcal{M}^0_{\text{per}} \cap \mathcal{M}_k \), and properties (P1), (P2), (P3), (P4), and (P5) hold with a \( g \)-minimal solution \( w(\cdot) \in C^2(\mathbb{R}^1) \), a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \) and integers \( m, n \). Then for each \( e \in (0, 1) \), there exists a neighborhood \( \mathcal{U} \) of \( g \) in \( \mathcal{M}_k \) such that for each \( h \in \mathcal{U} \) and each \( (h) \)-good function \( v \in W^{1,1}_\text{loc}([0, \infty)) \) there are integers \( p, q \) such that

\[
|v(t) - w(t + p) - q| \leq e \quad \text{for all large enough } t.
\]

(5.11)

**Lemma 5.5** [13, Corollary 6.1]. Assume that \( k \geq 3 \) is an integer, \( g \in \mathcal{M}^0_{\text{per}} \cap \mathcal{M}_k \), and properties (P1), (P2), (P3), (P4), and (P5) hold with a \( g \)-minimal solution \( w(\cdot) \in C^2(\mathbb{R}^1) \), a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \) and integers \( m, n \). Then there exist a neighborhood \( \mathcal{U} \) of \( g \) in \( \mathcal{M}_k \) and a number \( L > 0 \) such that for each \( h \in \mathcal{U} \) and each \( (h) \)-good function \( v \in W^{1,1}_\text{loc}([0, \infty)) \), the following property holds.

There is a number \( T_0 > 0 \) such that

\[
|v(t_2) - v(t_1) - \alpha(g)(t_2 - t_1)| \leq L
\]

(5.12)

for each \( t_1 \geq T_0 \) and each \( t_2 > t_1 \).
Completion of the proof of Theorem 3.1. Let \( k \geq 3 \) be an integer and let \( \alpha = \frac{mn}{n} \) be an irreducible fraction (\( n \geq 1 \) and \( m \) are integers). Let \( f \in \mathcal{M}_k \). By Proposition 2.2 there exists an \((f)\)-minimal solution \( w_f(\cdot) \in W_{1,1}^{1,1}(\mathbb{R}^1) \) such that
\[
w_f(t+n) = w_f(t) + m \quad \forall t \in \mathbb{R}^1.
\]
(5.13)

Choose
\[
\beta \in \partial E_f(\alpha).
\]
(5.14)

Consider a mapping \( \mathcal{A} : \mathcal{M}_k \to \mathcal{M}_k \) defined by (4.1). By Proposition 4.1 the mapping \( \mathcal{A} \) is continuous. Clearly there exists a continuous \( \mathcal{A}^{-1} : \mathcal{M}_k \to \mathcal{M}_k \).

Equations (5.14) and (4.18) imply that
\[
0 \in \partial E_{\mathcal{A}f}(\alpha), \quad E_{\mathcal{A}f}(\alpha) = \min \{ E_{\mathcal{A}f}(r) : r \in \mathbb{R}^1 \} = \mu(\mathcal{A}f)
\]
and that \( \mathcal{A}f \in \mathcal{M}_{\text{per}} \). It follows from Proposition 5.2 that there exists a bounded nonnegative function \( \phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2) \) such that
\[
\{ x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0 \} = \{ P(t, w_f(t)) : t \in [0, n] \}.
\]
(5.15)

Set \( f^{(\beta)} = \mathcal{A}f \) and for each \( \gamma \in (0, 1) \) define
\[
f_\gamma(t, x, u) = f(t, x, u) + \gamma \phi(P(t, x)), \quad (t, x, u) \in \mathbb{R}^3, \quad f_\gamma^{(\beta)} = \mathcal{A}(f_\gamma).
\]
(5.17)

Proposition 5.3 implies that for each \( \gamma \in (0, 1) \),
\[
f_\gamma^{(\beta)} \in \mathcal{M}_0^{\text{per}} \cap \mathcal{M}_k, \quad f_\gamma \longrightarrow f \quad \text{as} \quad \gamma \longrightarrow 0^+, \quad f_\gamma^{(\beta)} \longrightarrow f^{(\beta)} \quad \text{as} \quad \gamma \longrightarrow 0^+ \text{ in } \mathcal{M}_k.
\]
(5.18)

Fix \( \gamma \in (0, 1) \) and an integer \( n \geq 1 \). By Proposition 5.3 the properties (P1), (P2), (P3), (P4), and (P5) hold with \( g = f_\gamma^{(\beta)} \), \( a(g) = \alpha \) and \( w(\cdot) = w_f \).

By Lemmas 5.4 and 5.5, there exists an open neighborhood \( V(f, \gamma, n) \) of \( f_\gamma^{(\beta)} \) in \( \mathcal{M}_\gamma \) and a number \( L(f, \gamma, n) > 0 \) such that the following properties hold:

(i) for each \( h \in V(f, \gamma, n) \) and each \((h)\)-good function \( v \in W_{1,1}^{1,1}([0, \infty)) \), there are integers \( p, q \) such that
\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n}
\]
for all large enough \( t \);

(ii) for each \( h \in V(f, \gamma, n) \) and each \((h)\)-good function \( v \in W_{1,1}^{1,1}([0, \infty)) \), there is a number \( T_0 \) such that
\[
|v(t_2) - v(t_1) - \alpha(f_\gamma^{(\beta)})(t_2 - t_1)| \leq L
\]
for each \( t_1 \geq T_0 \) and each \( t_2 > t_1 \).
Let \( h \in V(f, y, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) be an \((h)\)-minimal solution with rotation number \( \alpha(h) \). Then by Proposition 2.3, (2.3), (5.4), and property (ii), \( v|_{[0,\infty)} \) is an \((h)\)-good function and there is \( T_0 \) such that (5.20) holds for each \( t_1 \geq T_0 \) and each \( t_2 > t_1 \). Since \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) has rotation number \( \alpha(h) \) it follows from Proposition 2.1 that there exists \( c_1 > 0 \) such that

\[
|v(t + s) - v(t) - \alpha(h)s| \leq c_1 \quad \forall s, t \in \mathbb{R}. \tag{5.21}
\]

Equations (5.15), (5.17), (5.20), and (5.21) imply that

\[
\alpha(h) = \alpha(f_{\beta}^y) = \alpha(f_{\beta}^i) = \alpha. \tag{5.22}
\]

Thus we have shown that

\[
\alpha(h) = \alpha \quad \forall h \in V(f, y, n). \tag{5.23}
\]

Let \( h \in V(f, y, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) be an \((h)\)-minimal solution with rotation number \( \alpha \). It follows from Proposition 2.3, (2.3), and (5.4) that \( v|_{[0,\infty)} \) is an \((h)\)-good function. By property (i) there exist integers \( p, q \) such that

\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \text{for all large enough } t. \tag{5.24}
\]

Therefore we proved the following property:

(iii) for each \( h \in V(f, y, n) \) and each \((h)\)-minimal solution \( v \in \mathcal{M}_h^{\text{per}}(\alpha) \), there exist integers \( p, q \) such that

\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \forall t \in \mathbb{R}^1. \tag{5.25}
\]

Define

\[
\mathcal{U}(f, y, n) = \mathcal{A}^{-1}(V(f, y, n)). \tag{5.26}
\]

Clearly \( \mathcal{U}(f, y, n) \) is an open neighborhood of \( f_y \) in \( \mathcal{M}_k \). By property (iii) the following property holds:

(iv) for each \( \xi \in \mathcal{U}(f, y, n) \) and each \((\xi)\)-minimal solution \( v \in \mathcal{M}_\xi^{\text{per}}(\alpha) \), there exist integers \( p, q \) such that (5.25) holds.

Define

\[
\mathcal{F}_{\kappa\alpha} = \cap_{n=1}^{\infty} \{ \mathcal{U}(f, y, i) : f \in \mathcal{M}_k, \ y \in (0, 1), \ i \geq n \}. \tag{5.27}
\]

It is not difficult to see that \( \mathcal{F}_{\kappa\alpha} \) is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \).
Let $g \in \mathcal{F}_{k\alpha}$, $e \in (0, 1)$ and $x, y \in \mathcal{M}^{(\text{per})}_{\alpha(\alpha)}$. Choose a natural number $n > 8e^{-1}$. By (5.27) there exist $f \in \mathcal{M}_k, \gamma \in (0, 1)$ and an integer $i \geq n$ such that

$$g \in \mathcal{U}(f, \gamma, i).$$

(5.28)

It follows from (5.28) and property (iv) that there exist integers $p_1, q_1, p_2, q_2$ such that

$$|x(t) - w_f(t + p_1) - q_1| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1,$$

(5.29)

$$|y(t) - w_f(t + p_2) - q_2| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1,$$

(5.30)

where $w_f \in \mathcal{M}^{(\text{per})}_f(\alpha)$.

It follows from (5.29) and (5.30) that for all $t \in \mathbb{R}^1$,

$$|x(t - p_1) - w_f(t) - q_1| \leq \frac{1}{i},$$

$$|y(t - p_2) - w_f(t) - q_2| \leq \frac{1}{i},$$

(5.31)

$$|x(t - p_1 - q_1) - (y(t - p_2) - q_2)| \leq \frac{2}{i},$$

$$|x(t + p_2 - p_1) - y(t) - q_1 + q_2| \leq \frac{2}{i} \leq \frac{2}{n} < e.$$

Since $e$ is any number in $(0, 1)$, we conclude that there exist integers $p, q$ such that

$$x(t + p) - q = y(t) \quad \forall t \in \mathbb{R}^1.$$

(5.32)

Assume that $h \in \mathcal{U}(f, \gamma, i)$ and $z \in \mathcal{M}^{(\text{per})}_h(\alpha)$. By the property (iv) there exist integers $p_3, q_3$ such that

$$|z(t) - w_f(t + p_3) - q_3| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1.$$

(5.33)

Combined with (5.29) this inequality implies that

$$|z(t - p_3) - q_3 - x(t - p_1) + q_1| \leq \frac{2}{i} \leq \frac{2}{n} < e$$

(5.34)

for all $t \in \mathbb{R}^1$. This completes the proof of Theorem 3.1.

References


Uniqueness of a minimal solution


**Alexander J. Zaslavski**: Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

*E-mail address: ajzasl@techunix.technion.ac.il*
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Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie